PT Symmetric

Quantum Mechanics

on

E2, E3 and E4

Euclidean Manifolds

Rudolph Kalveks

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PREFACE

I was drawn to the study of PT symmetric quantum mechanics by my curiosity to explore the boundaries of generally accepted quantum theory. I am indebted to Professor Carl Bender, who has been active in this field for many years, both for providing an initial induction into the topic and also for supervising my efforts to develop it in new directions. The proposal to extend the existing analyses of PT symmetric systems from configuration spaces described by the Heisenberg algebra to those described by the Euclidean algebras emerged from early discussions with Professor Chris Isham, who posed a number of penetrating questions. The subsequent development of the body of this work would not have been possible without the many insights and techniques associated with group theory, the geometry of spacetime manifolds and, of course, quantum theory, gleaned from the Imperial College Theoretical Physics faculty in the course of my MSc. I am thus grateful in particular to Professors Chris Hull, Amihay Hanany, Arkady Tseytlin, Joao Magueijo, Dan Waldram, Kelly Stelle and Doctors Toby Wiseman and Arttu Rajantie for facilitating a project that I hope others may also find of interest.

Rudolph Kalveks

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ABSTRACT

This study investigates PT symmetric quantum mechanics on the Euclidean Group E2, E3 and E4 manifolds, which embed the dynamics of the rotation groups SO(2), SO(3) and SO(4) on S^1, S^2 and S^3 spheres, respectively. Hilbert spaces on these manifolds are constructed based on finite group representations of their Lie Algebras, with the E3 representations studied including general spin representations. Non-Hermitian but PT symmetric ("PTS") Hamiltonians that act on these representations are constructed out of Lie Algebra operators. It is shown that Euclidean Group manifolds support PTS Hamiltonians with regions of unbroken PT symmetry, inside which all their eigenvalues are real and outside which their eigenvalues generally form complex conjugate pairs. It is found that the properties of PTS Hamiltonians can be related to the types of Lie Algebra operators used in their construction, with different characteristic behaviours attributable to raising or lowering operators, which are distinct from those of position or momentum measurement operators. It is further found that the presence of unbroken PT symmetry is not determined simply by the operator form of a given PTS Hamiltonian, but also depends on the representation chosen. Thus, for PTS Hamiltonians incorporating directional potentials based on position operators, zero spin periodic (bosonic) representations exhibit regions of unbroken PT symmetry while half integer spin antiperiodic (fermionic) representations do not. The critical values associated with coupling parameters that demarcate these regions are calculated. The group theoretic and quantum mechanisms underlying the differences between the PT symmetric behaviours of different operators and different representations are discussed.

Chapter One

Introduction

One of the central principles of conventional quantum mechanics is the characterisation of the energy and dynamics of systems in terms of Hermitian Hamiltonians. Under Schrödinger's Law, the state of a system with a Hermitian Hamiltonian evolves in a unitary manner and its norm is preserved over time. This happens because the Hermiticity of a Hamiltonian ensures the reality of the energy eigenvalues of the system. However, systems with non-Hermitian Hamiltonians can also have eigenstates with real eigenvalues and their evolution is similar to that of systems with Hermitian Hamiltonians. PT symmetric quantum mechanics explores the consequences of relaxing the conventional requirement of Hermiticity.

It has been shown (Bender, 2007) that non-Hermitian Hamiltonians that exhibit PT symmetry, defined as invariance under the combination of Parity and Time Reversal transformations, can have entirely real energy eigenvalues. What makes this phenomenon particularly intriguing is that such systems often contain parametric regions inside which all eigenvalues are real, surrounded by regions in which some eigenvalues become complex. The result is that whether a given state is harmonic and stable on the one hand, or dissipative and unstable on the other, depends on a simple coupling parameter, and such systems can undergo phase transitions if critical values for this parameter are crossed.

There is a growing body of experimental evidence that non-Hermitian but PT symmetric Hamiltonians provide a useful framework for understanding quantum mechanical systems in which critical points arise and phase transitions occur, such as

optical systems (Chong et al., 2011) and superconducting systems (Chtchelkatchev et al., 2010), (Rubinstein et al., 2007).

The main objective of this paper is to investigate the extent to which this behaviour of non-Hermitian but PT symmetric quantum mechanical systems can be expected to extend from one-dimensional systems to systems with configuration spaces described by higher dimensioned manifolds. The particular manifolds studied herein include those associated with the Euclidean E2, E3 and E4 groups, in other words, the circle S^1, sphere S^2 and three-sphere S^3, and their associated rotation groups SO(2), SO(3) and SO(4). The implementation of quantum mechanics on these manifolds follows the general approach (Isham, 1983) of identifying compatible conjugate operators based on geometric considerations and then using group theoretic methods to construct, from first principles, the possible representations of quantum states on the manifold, along with the Hamiltonians that may govern their evolution.

This Chapter contains a summary of the principles underlying PT symmetric quantum mechanics, largely following Bender (Bender, 2007). Chapter Two then investigates the PT symmetry of E2, building upon aspects of recent work in this area (Bender and Kalveks, 2010). Chapters Three and Four extend the investigation to the E3 and E4 groups. In the case of E3 the study covers general spin representations. It is found that non-Hermitian PT symmetric systems with entirely real eigenvalues can readily be constructed out of the elements of all these group algebras, with some important differences between the PT symmetric properties of states in different irreducible representations ("irreps"), such as bosonic and fermionic states of different spin.

Chapter 5 discusses the interpretation of the findings, using group theoretic considerations to shed light on the behaviour of states in different representations, and

draws conclusions about the conditions under which non-Hermitian Hamiltonians exhibit unbroken PT symmetry. The potential extension of the methodology to study the dynamics of non-Hermitian systems on more complicated spacetime manifolds is also discussed.

Principles of PT Symmetry

According to the Stone-Von Neumann theorem, the classical quantum mechanical relations on a $\mathbb{R}^n \times \mathbb{R}^n$ configuration space of n objects with positions $\hat{x}_i^\mu$ and momenta $\hat{p}_{vj}$,, where the index i extends from 1 to n, are uniquely given by the Heisenberg commutation relations (Isham, 1983):

$$\left[\hat{x}_i^\mu, \hat{p}_{vj}\right] = i\hbar\delta_v^\mu\delta_{ij}. \tag{1.1}$$

We shall henceforth work in natural coordinates such that c=1 and $\hbar = 1$.

We can define the improper Lorentz group operations of Parity and Time Reversal acting on the time dimension and one of the spatial dimensions of d+1 dimensional spacetime as the discontinuous transformations:

$$\begin{aligned} \mathcal{P}: &\quad x_i^\alpha \to -x_i^\alpha \ \text{ where } \alpha \in \{1\ldots d\}, \\ \mathcal{T}: &\quad x_i^0 \to -x_i^0. \end{aligned} \tag{1.2}$$

These Parity and Time Reversal operators satisfy the relations:

$$\begin{aligned} &\mathcal{P}^2 = \mathcal{T}^2 = I, \\ &[\mathcal{P}, \mathcal{T}] = 0. \end{aligned} \tag{1.3}$$

Classical quantum systems are described by Hermitian Hamiltonians that satisfy the Schrodinger equation:

$$\mathcal{H}|\psi\rangle = i\partial_t|\psi\rangle, \qquad \text{where } \mathcal{H} = \mathcal{H}^\dagger. \tag{1.4}$$

We can therefore equivalently associate the Time Reversal operator with the action of complex conjugation:

$$\begin{aligned} \boldsymbol{\mathcal{P}}: &\quad x_i^\alpha \to -x_i^\alpha \text{ for } \alpha \in \{1\ldots d\}, \\ \boldsymbol{\mathcal{T}}: &\quad i \to -i. \end{aligned} \tag{1.5}$$

In a representation where $\hat{p}_{vj} = -i\dfrac{\partial}{\partial x_j^v}$, the Heisenberg relations are invariant under the actions of $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{T}}$:

$$\begin{aligned} \boldsymbol{\mathcal{P}}: &\quad \hat{p}_{vj} \to -\hat{p}_{vj}, \\ &\quad \left[\hat{x}_i^\mu, \hat{p}_{vj}\right] = i\delta_v^\mu \delta_{ij} \to \left[-\hat{x}_i^\mu, -\hat{p}_{vj}\right] = \left[\hat{x}_i^\mu, \hat{p}_{vj}\right] = i\delta_v^\mu \delta_{ij}; \\ \boldsymbol{\mathcal{T}}: &\quad \hat{p}_{vj} \to -\hat{p}_{vj}, \\ &\quad \left[\hat{x}_i^\mu, \hat{p}_{vj}\right] = i\delta_v^\mu \delta_{ij} \to \left[\hat{x}_i^\mu, -\hat{p}_{vj}\right] = -i\delta_v^\mu \delta_{ij}; \ \left[\hat{x}_i^\mu, \hat{p}_{vj}\right] = i\delta_v^\mu \delta_{ij}. \end{aligned} \tag{1.6}$$

We can naturally expect these spacetime symmetries of the Heisenberg algebra to be manifested in the symmetries of quantum systems.

The Schrodinger equation (1.4) ensures that the evolution of a system is unitary and that the norm of a state is preserved over time:

$$\begin{aligned} &|\psi\rangle = e^{-i\mathcal{H}t}|\psi_0\rangle, \\ &\langle\psi|\psi\rangle = \langle\psi_0|e^{i\mathcal{H}^\dagger t}e^{-i\mathcal{H}t}|\psi_0\rangle = \langle\psi_0|\psi_0\rangle. \end{aligned} \tag{1.7}$$

In the case where $\boldsymbol{\mathcal{H}}$ is time independent, the spatial and temporal elements of states decouple and can be characterised by harmonic functions of constant energy:

$$\boldsymbol{\mathcal{H}}|\psi\rangle = \mathrm{E}|\psi\rangle = i\partial_t|\psi\rangle. \tag{1.8}$$

If we have a Hermitian Hamiltonian matrix that is not in diagonal form, it can be diagonalised by a unitary transformation.

We can also transform any Hermitian Hamiltonian to a non-Hermitian Hamiltonian by a similarity transformation. For example, working within a matrix formulation and using the similarity transformation $\mathcal{S} \in \mathrm{GL}(N,\mathbb{C})$, where $\mathcal{S}$ is non-unitary and N is the dimension of the Hilbert space, we can consider the non-Hermitian Hamiltonian $\mathcal{N}$ and its associated basis states $|\Omega\rangle$ given by:

$$\begin{aligned} \mathcal{N} &\equiv \mathcal{S}\mathcal{H}\mathcal{S}^{-1}, \\ |\Omega\rangle &= \mathcal{S}|\psi\rangle. \end{aligned} \tag{1.9}$$

The eigenvalues of any matrix are unaffected by a similarity transformation, and so, even though it is not generally the case that non-Hermitian Hamiltonians have real eigenvalues, $\mathcal{N}$ diagonalises to the same eigenvalues as $\mathcal{H}$. However, the basis states $|\Omega\rangle$ do not have the same norm as the $|\psi\rangle$ states since $\mathcal{S}^{-1\dagger}\mathcal{S}^{-1} \neq I$ if $\mathcal{S}$ is non-unitary:

$$\langle\psi|\psi\rangle = \underbrace{\langle\Omega|\mathcal{S}^{-1\dagger}\mathcal{S}^{-1}}_{\langle\tilde{\Omega}|}|\Omega\rangle, \text{ where } \langle\Omega| \equiv |\Omega\rangle^{\dagger}. \tag{1.10}$$

In order to obtain the correct norm in the $\mathcal{N}$ basis, we must define the inner product using $\langle\tilde{\Omega}|$ from (1.10). This is analogous to using a metric on the Hilbert space to lower the Hermitian conjugate $\langle\Omega|$ before taking the inner product. The inconvenience is that wavefunctions in the $\mathcal{N}$ basis are not self-adjoint with respect to the inner product and calculation of $\langle\tilde{\Omega}|$ from $|\Omega\rangle$ requires knowledge of the entire Hilbert space.

If we consider the time development of this redefined norm we have:

$$\begin{aligned}\left\langle \tilde{\Omega}_t \middle| \Omega_t \right\rangle &= \left\langle \Omega_t \middle| \mathcal{S}^{-1\dagger}\mathcal{S}^{-1} \middle| \Omega_t \right\rangle \\ &= \left\langle \Omega_0 \middle| e^{i\mathcal{N}^\dagger t}\mathcal{S}^{-1\dagger}\mathcal{S}^{-1}e^{-i\mathcal{N}t} \middle| \Omega_0 \right\rangle \\ &= \left\langle \Omega_0 \middle| e^{i\mathcal{S}^{-1\dagger}\mathcal{H}^\dagger\mathcal{S}^\dagger t}\mathcal{S}^{-1\dagger}\mathcal{S}^{-1}e^{-i\mathcal{S}\mathcal{H}\mathcal{S}^{-1}t} \middle| \Omega_0 \right\rangle \\ &= \left\langle \Omega_0 \middle| \mathcal{S}^{-1\dagger}e^{i\mathcal{H}^\dagger t}\mathcal{S}^\dagger\mathcal{S}^{-1\dagger}\mathcal{S}^{-1}\mathcal{S}e^{-i\mathcal{H}t}\mathcal{S}^{-1} \middle| \Omega_0 \right\rangle \\ &= \left\langle \Omega_0 \middle| \mathcal{S}^{-1\dagger}\mathcal{S}^{-1} \middle| \Omega_0 \right\rangle \\ &= \left\langle \tilde{\Omega}_0 \middle| \Omega_0 \right\rangle,\end{aligned} \tag{1.11}$$

and so the redefined norm of $|\Omega\rangle$ is preserved over time by the underlying unitary evolution.

The foregoing considerations apply to any non-Hermitian Hamiltonian that is related to a Hermitian Hamiltonian by some similarity transformation. We can now specialise to the case where $\mathcal{N}$ is PT symmetric, which means that:

$$\left[\mathcal{N}, \mathcal{PT}\right] = 0. \tag{1.12}$$

This entails a corresponding symmetry in $\mathcal{H}$ that may be expressed in terms of a new operator $\mathcal{C}$:

$$\begin{aligned}&\left[\mathcal{N}, \mathcal{PT}\right] = 0, \\ &\left[\mathcal{S}\mathcal{H}\mathcal{S}^{-1}, \mathcal{PT}\right] = 0, \\ &\left[\mathcal{H}, \mathcal{S}^{-1}\mathcal{PTS}\right] = 0, \\ &\left[\mathcal{H}, \mathcal{C}\right] = 0, \text{ where } \mathcal{C} \equiv \mathcal{S}^{-1}\mathcal{PTS}.\end{aligned} \tag{1.13}$$

From (1.3) and (1.13) we can show that the $\mathcal{C}$ operator always satisfies $\mathcal{C}^2 = I$. The $\mathcal{C}$ operator is not unique since $\mathcal{N}$ can be diagonalised in multiple ways given by the permutation group S_N. Other properties of the $\mathcal{C}$ operator have also been examined (Bender and Mannheim, 2009), (Bender et al., 2004), (Bender and Jones, 2004), (Bender and Klevansky, 2009).

We shall henceforth refer to all *non-Hermitian PT symmetric* Hamiltonians simply as *PT symmetric* ("PTS"). Following established terminology (Bender, 2007), PT symmetric Hamiltonians are said to exhibit *unbroken PT symmetry* if all their eigenvalues are real. This condition is much more restrictive than that of simple PT symmetry and there are many PT symmetric Hamiltonians that have one or more complex eigenvalues. This latter category are said to exhibit *broken PT symmetry*. Certain PT symmetric Hamiltonians, including some of those studied in this paper, exhibit parametric regions (for some coupling constants) within which PT symmetry is broken as well as parametric regions within which PT symmetry is unbroken.

Group Theoretic and Topological Approaches to Quantisation

The group theoretic approach to quantisation starts from the Lie Algebra relations between quantum mechanical operators. Once a Lie Algebra is known, a maximal (Cartan) sub-algebra of commuting operators can be selected to provide a basis for describing and labelling quantum states of the system. The other operators within the Lie Algebra can in principle be arranged into raising and lowering operators acting upon these basis states. It then turns out that the spacetime symmetry properties of Hamiltonians constructed from these operators can be related to different operator roles within the Lie Algebra.

Geometric considerations (Isham) require that momentum operators defined within a Lie Algebra should behave as dynamic vectors and generate integral curves or flows over a manifold. This in turn places a constraint on the permissible coordinate form of momentum operators.

For example, if we take our manifold as the real line $\mathbb{R}$, with position operator $\hat{x}|\psi\rangle = x|\psi\rangle$, we can choose our Hermitian momentum operator to be $\hat{p} = -i\partial/\partial x$ and then the action of the unitary operator $\exp(-ia\hat{p})$ on x, where {a, x}$\in \mathbb{R}$, produces a displaced value x – a $\in \mathbb{R}$. The flow remains on the manifold and we can therefore use this momentum operator to formulate consistent commutation relations as $[\hat{x},\hat{p}] = i$. However, if we take our manifold as the half-line $\mathbb{R}+$, parameterised by y, then the flow from exponentiation of the operator $\hat{p} = -i\partial/\partial y$ could lead to y – a $\in \mathbb{R}-$, which is off the $\mathbb{R}+$ manifold (Isham). We have to define momentum on $\mathbb{R}+$ as $\hat{p} = -iy\partial/\partial y$ to ensure flows remain on the manifold. We then obtain the less familiar form of commutation relations $[\hat{y},\hat{p}] = i\hat{y}$.

It can be observed that $\{x,\mathbb{R}\}$ and $\{y,\mathbb{R}+\}$ are related by the diffeomorphism y=exp(x) and we can infer that once we have a consistent flow over some manifold, we may relate this to a consistent flow on some other manifold, providing their coordinate systems can be linked by continuous smooth coordinate transformation or diffeomorphism. The resulting form of the canonical commutation relations may however look very different to the Heisenberg relations (1.1) ruling within $\mathbb{R}^n$.

Given a set of operators that are compatible with regard to some manifold, the Lie Algebra commutation relations between them can be determined and we can then explore the representation theory of the group, identifying irreps and complete sets of basis functions and their relationships. It is conventionally preferable to work with finite irreps that close under the action of the raising and lowering operators of the Lie Algebra.

Chapter Two

PT Symmetry on E2

Geometric Definition

E2 is defined by the dynamics of the rotation group on the one sphere $S^1 \otimes SO(2)$. Following the geometric approach outlined above, we can establish consistent quantum mechanical operator relations on the S^1 manifold, by pulling these back from the canonical Heisenberg relations (1.1) on $\mathbb{R}^2$, using an embedding of S^1 in $\mathbb{R}^2$ as a circle of unit radius. In order to parameterise position continuously on S^1, we use two orthogonal coordinates and we define their respective operators as $\hat{u}$ and $\hat{v}$. We obtain the momentum operator $\hat{J}$ on S^1 by pulling back the angular momentum operator from $\mathbb{R}^2$. The key relationships in this mapping of operators to their angular coordinate representation are:

$$\begin{aligned}
&\hat{v} = x\big|_{r=1} = r\cos\theta\big|_{r=1} = \cos\theta, \\
&\hat{u} = y\big|_{r=1} = r\sin\theta\big|_{r=1} = \sin\theta, \\
&\hat{J} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = r\cos\theta\left(-i\partial_y\right) - r\sin\theta\left(-i\partial_x\right) = -i\left(\frac{\partial y}{\partial\theta}\partial_y + \frac{\partial x}{\partial\theta}\partial_x\right) = -i\partial_\theta .
\end{aligned} \tag{2.1}$$

We can verify that the operator $\hat{J}$ is Hermitian and also that it exponentiates to give a unitary operator that generates a flow over the manifold in the form of a rotation:

$$\exp\left(i\alpha\hat{J}\right)\begin{pmatrix}\hat{v}\\ \hat{u}\end{pmatrix} = \exp\left(\alpha\frac{\partial}{\partial\theta}\right)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\hat{v}\\ \hat{u}\end{pmatrix}. \tag{2.2}$$

Lie Algebra

The Lie Algebra of E2 (Isham, 1983) is given by these position and momentum operators on S^1 and is readily verified to be:

$$\begin{aligned} \left[\hat{u},\hat{J}\right] &= i\hat{v}, \\ \left[\hat{v},\hat{J}\right] &= -i\hat{u}, \\ \left[\hat{u},\hat{v}\right] &= 0. \end{aligned} \tag{2.3}$$

As Isham notes, we recover the Heisenberg relations (1.1) in the limit of small displacements:

$$\begin{aligned} \hat{u} &\to \delta\hat{\theta}, \\ \hat{v} &\to 1, \\ \left[\hat{J},\hat{u}\right] = -i\hat{v} &\to \left[\delta\hat{\theta},\hat{J}\right] = i. \end{aligned} \tag{2.4}$$

The E2 group contains an Abelian ideal and is thus not simple or semi-simple. Its Lie Algebra has a quadratic Casimir operator $\hat{C}$ [1], which commutes with all the other operators, and which can be interpreted in terms of the unit radius of the embedding circle:

$$\begin{aligned} \hat{C} &= \hat{u}^2 + \hat{v}^2 = I, \\ \left[\hat{C},\hat{u}\right] &= \left[\hat{C},\hat{v}\right] = \left[\hat{C},\hat{J}\right] = 0. \end{aligned} \tag{2.5}$$

Quantum states of S^1 can thus be characterised by the radius of the embedding circle and one of the three non-commuting operators, which we choose to be the momentum operator $\hat{J}$ so that we can describe dynamics. The position operators can be combined

[1] The Casimir $\hat{C}$ is unrelated to the $\mathcal{C}$ symmetry operator discussed in Chapter One.

into raising and lowering operators $\hat{W}_{\pm}$ which increment or decrement the angular momentum eigenvalue by an integer:

$$\begin{aligned} &\hat{W}_{\pm} = \hat{v} \pm i\hat{u}, \\ &\left[\hat{W}_{+}, \hat{W}_{-}\right] = 0, \\ &\left[\hat{J}, \hat{W}_{\pm}\right] = \hat{W}_{\pm}. \end{aligned} \tag{2.6}$$

We can now observe that, while $\hat{J}$ is a Hermitian operator, neither of the raising and lowering operators are Hermitian. However, if we take Parity as the reflection $\mathcal{P}:(\hat{u},\hat{v}) \to (-\hat{u},\hat{v})$, then we see that $\hat{W}_{+}$ and $\hat{W}_{-}$ are both PT symmetric. This non-Hermitian, but PT symmetric nature of raising and lowering operators plays a key role in determining the behaviour of the eigenstates of PT symmetric Hamiltonians.

Basis for Eigenstates

The momentum operator has its eigenstates given by the irreps of SO(2) or U(1), which is the symmetry group of S^1. Thus, $\hat{J}$ has an infinite sequence of eigenstates, which are harmonic with respect to θ, and which have real eigenvalues:

$$\begin{aligned} &\hat{J}\left|\psi_m\right\rangle = -i\partial_\theta \left|\psi_m\right\rangle = m\left|\psi_m\right\rangle \ , \\ &\left|\psi_m\right\rangle \propto e^{im\theta}. \end{aligned} \tag{2.7}$$

The condition of periodicity under rotation by 2π leads to the constraint that *m* should take positive or negative integer values. Following Bender (Bender and Kalveks, 2010), we shall relax this condition and also consider antiperiodic states for which *m* takes half-integer values. The normalised free eigenstates $\left|\psi_m\right\rangle = \frac{1}{\sqrt{2\pi}} e^{im\theta}$, where $m = -\infty,\ldots 0, \frac{1}{2}, 1, \ldots \infty$, then form a complete basis for the Fourier decomposition of a

state into its periodic (bosonic) and antiperiodic (fermionic) components. Choosing Parity as the reflection $\mathcal{P}:(\hat{u},\hat{v})\to(-\hat{u},\hat{v})\Leftrightarrow\theta\to-\theta$, the basis states have the spacetime symmetry transformation properties:

$$\begin{aligned}
\mathcal{P}: &\quad |\psi_m\rangle = e^{im\theta} \to e^{im(-\theta)} = |\psi_{-m}\rangle,\\
\mathcal{T}: &\quad |\psi_m\rangle = e^{im\theta} \to e^{(-i)m\theta} = |\psi_{-m}\rangle,\\
\mathcal{PT}: &\quad |\psi_m\rangle = e^{im\theta} \to e^{(-i)m(-\theta)} = |\psi_m\rangle.
\end{aligned} \tag{2.8}$$

Thus, while $\mathcal{P}$ and $\mathcal{T}$ taken individually reverse the sign of m, we have the result that the eigenstates of the Hermitian operator $\hat{J}$ are PT symmetric.

Hermitian and PT Symmetric Hamiltonians

We can form a Hamiltonian over S^1 from the combination of the momentum operator (squared to achieve dimensions consistent with energy) with a potential $\hat{V}(\hat{u},\hat{v})$. For example, to construct a Hermitian Hamiltonian, we can set $\hat{V} = g\hat{v}$, where g is some constant real coupling parameter, and obtain:

$$\hat{J}^2|\psi\rangle + g\hat{v}|\psi\rangle = E|\psi\rangle\ . \tag{2.9}$$

This Hamiltonian is related by a simple transformation of coordinates and redefinition of parameters to the Mathieu equation (Bender and Orszag, 1978):

$$\frac{d^2y}{dt^2} + (a + 2\varepsilon\cos t)y = 0. \tag{2.10}$$

The solutions to the Mathieu equation have various well-defined properties, which map to properties of the eigenstates and eigenvalues of the Hamiltonian (2.9). For any value of the parameter g, two infinite series of solutions to (2.9) exist: symmetric solutions $C_n(g)$ and anti-symmetric solutions $S_n(g)$. The eigenstates in each series alternate

between being periodic and antiperiodic and can be labelled by a number n, which takes integer values for periodic solutions and half-integer values for antiperiodic solutions. For n=0, there are only symmetric eigenstates $C_0(g)$.

If the potential vanishes (g=0), the symmetric and the anti-symmetric eigenstates are degenerate in terms of E and we recover the free eigenstate basis (2.7) with the eigenvalue E given by m^2. For this case, the symmetric and antisymmetric Mathieu solutions reduce to the cosine and sine series respectively and so the free eigenstate basis (2.7) is related to the Mathieu equation solutions (up to normalisation) by:

$$\left|\psi_m\right\rangle = C_m(0) + iS_m(0). \tag{2.11}$$

For g non-zero and real, the symmetric and anti-symmetric eigenstates of (2.9) become non-degenerate and their eigenvalues split. Importantly, however, the eigenvalue E remains real for all real values of g, as is to be expected from a Hermitian Hamiltonian.

Interestingly, we can use this latter property to derive a set of non-Hermitian Hamiltonians that also have entirely real eigenvalues. The implicit assumption of traditional quantum mechanics is that coordinates are integrated along their real axes (Bender, 2007). This choice of contour, along with explicit or implicit boundary conditions, establishes the constraints necessary to solve the wave equation and to determine the energy spectrum. In the case of the 2D system we are considering, the appropriate boundary condition is one of periodicity in 2π and this defines the wavefunction solution in the case that the contour along which the eigenstates are integrated is the real θ axis. We can however consider the consequences of selecting a different contour of integration.

A simple way of doing this is to displace the coordinate system along the imaginary axis, while retaining the requirement for periodicity over 2π along the real axis. This is achieved by the coordinate and parameter substitutions $\theta \to \theta + i\alpha$ and $g \to g/\cosh\alpha$, for some arbitrary fixed real α. Under these substitutions, equation (2.9) becomes:

$$\hat{J}^2\psi + g\left(\hat{v} - i\,\hat{u}\,\tanh\alpha\right)\psi = E\psi. \qquad (2.12)$$

Since we have only made a coordinate substitution, the eigenfunctions retain their eigenvalues E, which therefore remain real.

This is now no longer a Hermitian Hamiltonian, but rather a PT symmetric Hamiltonian, under Parity chosen as $\mathcal{P}:(\hat{u},\hat{v}) \to (-\hat{u},\hat{v})$. This Hamiltonian is a representative of a class of PT symmetric Hamiltonians for arbitrary real g and real α. The limits $|\tanh\alpha| \le 1$ bound a parametric region inside which this PT symmetric Hamiltonian has all real eigenvalues. For the limiting cases $\tanh\alpha = \pm 1$, the potential becomes $\hat{V}(\hat{u},\hat{v}) = g\hat{W}_{\pm}$ and so we see that the raising and lowering operators, when incorporated into simple potentials, give rise to unbounded regions of unbroken PT symmetry. (For values of $\tanh\alpha > 1$ (α complex), the eigenvalues are no longer real.)

We can also adopt a direct approach to exploring the reality of the eigenvalues of a PT symmetric form of (2.9) by taking the coupling parameter to be imaginary. This leads to a Hamiltonian that is PT symmetric under Parity chosen as $\mathcal{P}:(\hat{u},\hat{v}) \to (\hat{u},-\hat{v})$:

$$\hat{J}^2\psi + i\,\mathrm{Re}(g)\,\hat{v}\psi = E\psi\ . \qquad (2.13)$$

If we wish to understand the behaviour of eigenvalues under this imaginary potential, we cannot just draw upon the general properties of Mathieu functions, but rather need to investigate numeric solutions to the equations.

Solution Methodology: Mathieu Functions vs Matrix Diagonalisation

The Mathieu equation solutions are implemented within *Mathematica* and this gives one route for obtaining eigenstates and their eigenvalues for arbitrary complex values of the coupling parameter g. To achieve generality and to obtain greater insight into the splitting and mixing of eigenstates, we can alternatively work within a matrix mechanics framework. Both methods give numerically equal results.

Given a general potential $\hat{V}(\hat{u},\hat{v})$, suppose we require the solutions to the equation:

$$\hat{J}^2|\psi_E\rangle + \hat{V}|\psi_E\rangle = E|\psi_E\rangle. \tag{2.14}$$

In the absence of a general analytic solution, we may proceed by expanding the solution in terms of the free eigenstate basis:

$$|\psi_E\rangle = \sum_m a_m |\psi_m\rangle. \tag{2.15}$$

We can then rearrange (2.14) and restate the Hamiltonian relative to the eigenstate basis using the completeness and orthonormality properties of the basis to obtain:

$$\sum_m \underbrace{\langle\psi_{m'}|m^2 + \hat{V}(\hat{u},\hat{v})|\psi_m\rangle}_{Hamiltonian} a_m = E\ a_{m'}. \tag{2.16}$$

The resulting relationships are in the form of a typical eigenvalue matrix equation $\mathcal{H}\mathbf{a} = E\mathbf{a}$, albeit over a vector space of infinite dimension. While this may make an exact solution intractable, we can investigate whether numeric eigenvalue solutions converge acceptably when a finite subset of eigenvalues is diagonalised (i.e. within a finite m×m matrix for some integer m). This is a version of the Galerkin method (Bender and Orszag, 1978).

In the case where $\hat{V}(\hat{u},\hat{v}) = g\hat{v} = g\cos\theta$, we can evaluate the matrix elements analytically:

$$\langle \psi_{m'} | \hat{V}(\hat{u},\hat{v}) | \psi_m \rangle = \frac{g}{2\pi} \int e^{i(m-m')\theta} \cos\theta \, d\theta = \frac{1}{2} g \left(\delta_{m'-1,m} + \delta_{m'+1,m} \right) \qquad (2.17)$$

$$\sum_m \left[m^2 \delta_{m'm} + \frac{1}{2} g \left(\delta_{m'-1,m} + \delta_{m'+1,m} \right) \right] a_m = E \; a_{m'} \qquad (2.18)$$

The g cosθ potential, being a combination of the raising and lowering operators, only mixes immediately adjacent bosonic or immediately adjacent fermionic states separated by a single integer m value. Thus, for the periodic series given by integer values of m, we obtain the Hamiltonian matrix:

$\langle m' \vert \mathcal{H} \vert m \rangle$	$\vert -m \rangle$	…	$\vert -1 \rangle$	$\vert 0 \rangle$	$\vert 1 \rangle$	…	$\vert m \rangle$
$\langle -m \vert$	m^2	$g/2$	0	0	0	0	0
	$g/2$	…	$g/2$	0	0	0	0
$\langle -1 \vert$	0	$g/2$	1	$g/2$	0	0	0
$\langle 0 \vert$	0	0	$g/2$	0	$g/2$	0	0
$\langle 1 \vert$	0	0	0	$g/2$	1	$g/2$	0
	0	0	0	0	$g/2$	…	$g/2$
$\langle m \vert$	0	0	0	0	0	$g/2$	m^2

. (2.19)

It will be recalled that the trace of a matrix is invariant under diagonalisation, with the degree of mixing between diagonal terms dependent on the coupling introduced by the off diagonal elements. One consequence of the *symmetric tridiagonal* form, combined with the relative scaling between the diagonal terms (increasing as m^2) and their adjacent elements (fixed at g/2) is that relatively little mixing occurs between distant diagonal terms. Thus, for small values of the parameter g, a very good approximation to the first few eigenvalues can be obtained with a matrix of a manageable size. The first

six fermionic and bosonic eigenvalues in the next section were obtained using a 12 x 12 matrix, at which point numeric convergence had occurred to 4 significant figures for the range of potentials considered. The eigenstates of the system determined by this diagonalisation process are superpositions of the basis states and correspond to the symmetric and antisymmetric Mathieu functions.

Eigenvalue Splitting Patterns: $g\hat{v}$ or $g\cos\theta$ potential

The patterns of eigenvalue splitting for the first few eigenstates in the bosonic and fermionic series under a $g\cos\theta$ potential, for real and imaginary values of the coupling parameter, are shown in Figures 1 to 6.

For g non-zero and real, the symmetric and anti-symmetric functions become non-degenerate and their eigenvalues split, with the degree of splitting increasing with g, but reducing with m. The eigenvalues are real, although at least the first one takes a negative value.

Intriguingly, even though complex values of the parameter g invariably lead to complex eigenvalues (not shown), imaginary values of g can support real eigenvalues in certain circumstances.

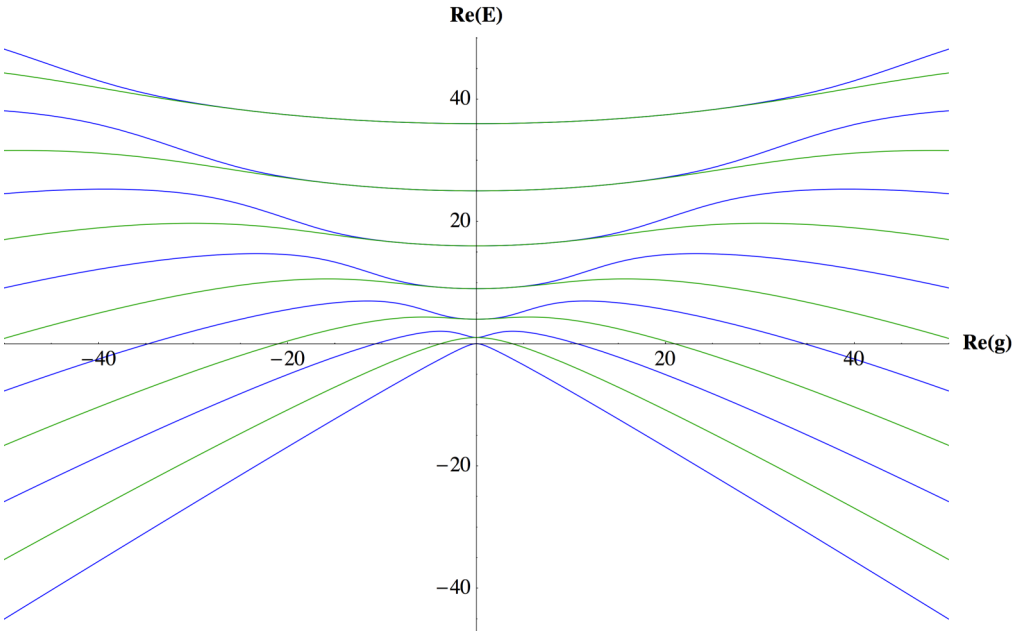

Figure 1. Bosonic eigenvalues for real values of g in the range (–50,50). The blue curves represent symmetric eigenfunctions m=0 to 6 and the green curves represent antisymmetric eigenfunctions m=1 to 6. The E axis intercepts give m^2 eigenvalues.

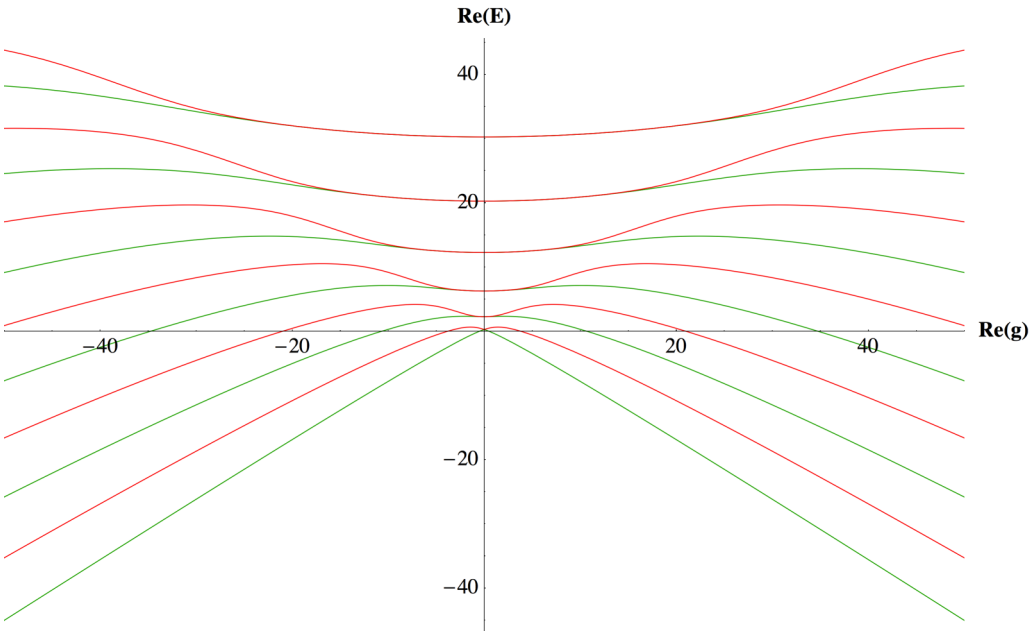

Figure 2. Fermionic eigenvalues for real values of g in the range (–50,50). The red curves represent symmetric eigenfunctions m=1/2 to 11/2 and the green curves represent antisymmetric eigenfunctions m=1/2 to 11/2. The E axis intercepts give m2 eigenvalues.

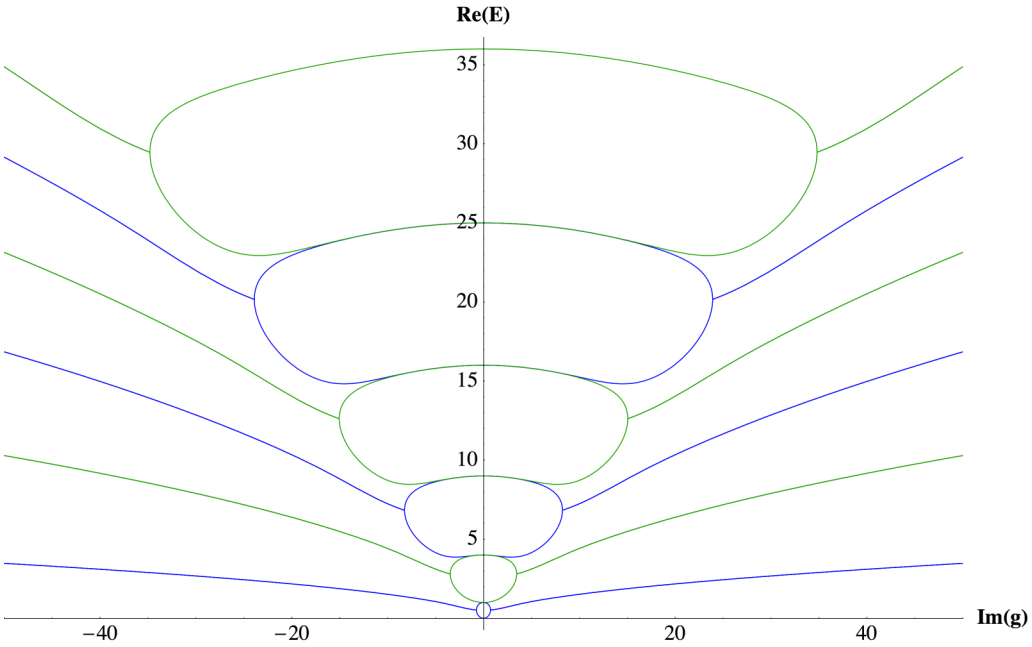

Figure 3. Real components of bosonic eigenvalues for values of g in the range (–50i, 50i). The blue curves represent symmetric eigenfunctions m=0 to 5 and the green curves represent antisymmetric eigenfunctions m=1 to 6. The E axis intercepts give m2 eigenvalues.

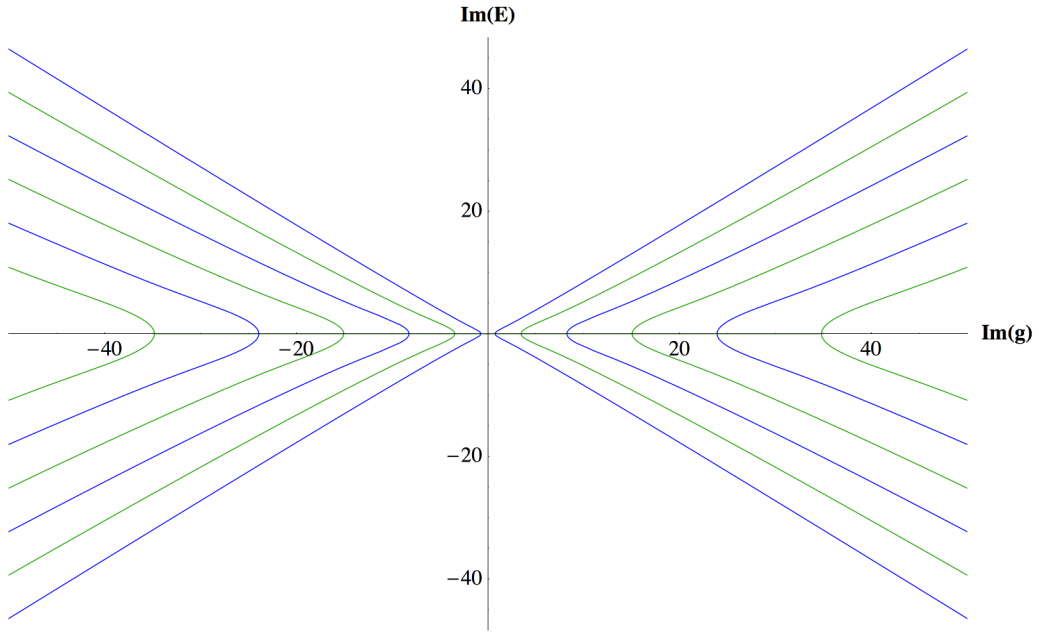

Figure 4. Imaginary components of bosonic eigenvalues for values of g in the range (–50i, 50i). The blue curves represent conjugate pairs for symmetric eigenfunctions m=0 to 5 and the green curves represent conjugate pairs for antisymmetric eigenfunctions m=1 to 6.

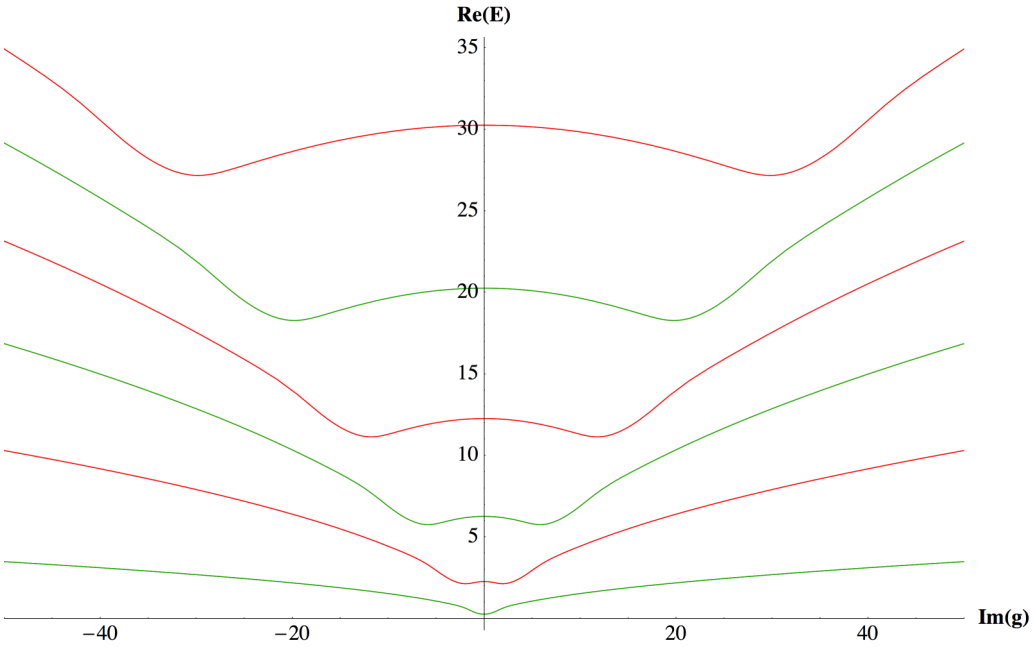

Figure 5. Real components of fermionic eigenvalues for values of g in the range (–50i, 50i). The curves represent eigenfunctions m=1/2 to 11/2. The E axis intercepts give m2 eigenvalues.

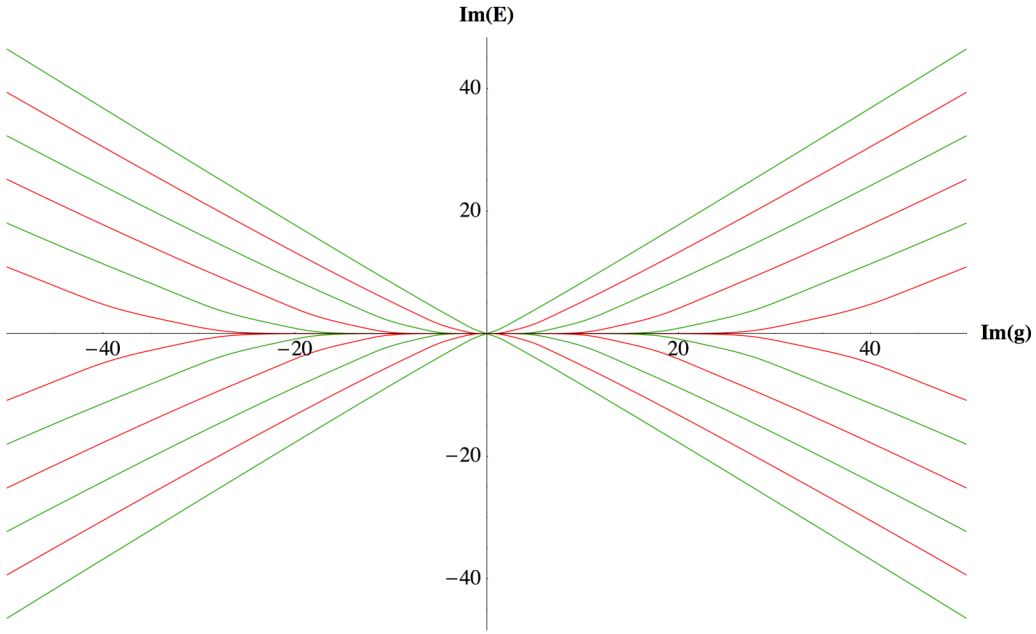

Figure 6. Imaginary components of fermionic eigenvalues for values of g in the range (–50i, 50i). The curves represent conjugate pairs for eigenfunctions m=1/2 to 11/2. There is no region of unbroken PT symmetry.

Several points are noteworthy:

(1) For real values of g, both the bosonic and fermionic series have real eigenvalues throughout.

   a. The bosonic eigenvalues are shifted from the m^2 values {1, 4, 9, 16, 25, 36…} ruling at g=0 and the degeneracy between the symmetric and antisymmetric series for m>0 is broken with the symmetric series being shifted upwards relative to the antisymmetric series.

   b. The fermionic eigenvalues are similarly shifted from the m^2 values {1/4, 9/4, 25/4, 49/4, 81/4, 121/4…} ruling at g=0 and the degeneracy between the symmetric and antisymmetric series is broken with the symmetric series being shifted upwards relative to the antisymmetric series.

   c. The ground state with m=0 has its E eigenvalue shifted below zero.

(2) For imaginary values of g, the behaviours of the bosonic and fermionic series differ markedly.

   a. In the case of the bosonic series, the eigenvalues remain real for small imaginary values of g, but as g increases, each adjacent pair converges to a critical real value, beyond which any further increase in g gives rise to a complex conjugate pair. The symmetric and antisymmetric series combine separately into these complex conjugate pairs.

   b. In the case of the fermionic series, an arbitrarily small imaginary value of g gives rise to a complex conjugate pair. The antisymmetric and symmetric series remain degenerate.

The critical imaginary values of g at which splitting into conjugate pairs occurs for each eigenstate pair are summarised in the table below. Below the first critical value, all the bosonic eigenstates of the PT symmetric Hamiltonian (2.9) enjoy a region of unbroken PT symmetry. The fermionic eigenstates do not exhibit regions of unbroken PT symmetry. This difference in behaviour appears to be related to the different structure of the fermionic Hamiltonian matrix, in which the unperturbed eigenstates all start off as degenerate pairs since there is no m=0 state.

Eigenstate Pairs $\{m_1, m_2\}$	*Series*	*Critical Value* g
$\{0,1\}$	*Symmetric*	0.7344 I
$\{1,2\}$	*Antisymmetric*	3.4645 I
$\{2,3\}$	*Symmetric*	8.2356 I
$\{3,4\}$	*Antisymmetric*	15.0484 I
$\{4,5\}$	*Symmetric*	23.9030 I
$\{5,6\}$	*Antisymmetric*	34.7994 I

(2.20)

Other potentials

It is natural to ask how the behaviour of potentials based on other elements of the Lie Algebra compares. By S^1 symmetry, a $\hat{u}$ potential is similar to $\hat{v}$.

However, if we work with just the raising operator $\hat{V} = g\hat{W}_+$, then the Hamiltonian matrix takes the form $\mathcal{H} = m^2\delta_{m'm} + \frac{1}{2}g\delta_{m'+1,m}$, which is upper triangular and diagonalises to give real m^2 eigenvalues that are unchanged from the basis states, regardless of the value of g (although the basis states are mixed into superpositions by the potential). Thus the raising and lowering operators $W_\pm$ both give rise to unbounded regions of unbroken PT symmetry for both bosonic and fermionic states.

Finally we can consider a potential $V = g\hat{J}$. This leads to a Hamiltonian matrix of the form $\mathcal{H} = m^2\delta_{m'm} + gm\delta_{m',m}$, which is Hermitian if g is real or is PT symmetric if g is imaginary. However, the Hamiltonian is already in diagonal form, with the result that if g is imaginary, then all the eigenvalues are complex (except for m=0), and so neither the bosonic nor the fermionic states have a region of unbroken PT symmetry.

Key Findings: E2

Quantum mechanics on the $S^1 \otimes SO(2)$ manifold, characterised by the E2 group, contains non-Hermitian, but PT symmetric Hamiltonians. These include Hamiltonians of the form:

$$\begin{aligned}
&\hat{\mathcal{H}}(\hat{u},\hat{v}) = \hat{J}^2 + g(\hat{v} - i\,\hat{u}\tanh\alpha) && (g \text{ and } \alpha \text{ real}),\\
&\hat{\mathcal{H}}\left(\hat{W}_{\pm}\right) = \hat{J}^2 + g\hat{W}_{\pm} && (g \text{ real or imaginary}),\\
&\hat{\mathcal{H}}(\hat{v}) = \hat{J}^2 + g\hat{v} && (g \text{ imaginary}),\\
&\hat{\mathcal{H}}(\hat{u}) = \hat{J}^2 + g\hat{u} && (g \text{ imaginary}).
\end{aligned} \tag{2.21}$$

In the cases of $\hat{\mathcal{H}}(\hat{u},\hat{v})$ and $\hat{\mathcal{H}}\left(\hat{W}_{\pm}\right)$, we have shown that unbounded regions of unbroken PT symmetry exist, regardless of whether eigenfunctions are bosonic (periodic over 2π) or fermionic (antiperiodic over 2π). In the cases of $\hat{\mathcal{H}}(\hat{v})$ and $\hat{\mathcal{H}}(\hat{u})$, we have shown that regions of unbroken PT symmetry exist only for the bosonic series for values of g below a critical value. These Hamiltonians can naturally be generalised using the rotational symmetry of S^1 and should be viewed as representatives of equivalence classes.

Chapter 3

PT Symmetry on E3

Geometric Definition

Quantum mechanics on E3 can be viewed as a generalisation from S^1 to S^2 and the wavefunctions of E3 systems are characterised in terms of the dynamics of the rotation group on the two sphere $S^2 \otimes SO(3)$. Following the geometric approach, we obtain position and momentum operators and their algebra by pulling back the Heisenberg relations (1.1) from Cartesian coordinates over $\mathbb{R}^3$ onto the surface of an S^2 sphere of unit radius. In order to parameterise position continuously on S^2, we need to use three orthogonal coordinates and we define the respective operators as $\hat{u}$, $\hat{v}$ and $\hat{w}$. We obtain three momentum operators $\hat{J}_u, \hat{J}_v$ and $\hat{J}_w$ on S^2 by pulling back the orbital angular momentum operators from $\mathbb{R}^3$. The key relationships in this mapping are:

$$\begin{aligned} \hat{u} &= x|_{r=1} = r\sin\theta\cos\phi|_{r=1} = \sin\theta\cos\phi, \\ \hat{v} &= y|_{r=1} = r\sin\theta\sin\phi|_{r=1} = \sin\theta\sin\phi, \\ \hat{w} &= z|_{r=1} = r\cos\theta|_{r=1} = \cos\theta. \end{aligned} \tag{3.1}$$

$$\begin{aligned} \hat{J}_u &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = i\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right), \\ \hat{J}_v &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) = i\left(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\sin\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right), \\ \hat{J}_w &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\phi}. \end{aligned} \tag{3.2}$$

As in the case of E2, we can verify that the angular momentum operators are Hermitian and that they exponentiate to give unitary operators that generate flows over S^2 in the form of rotations, for example:

$$\exp\left(i\alpha\hat{J}_w\right)\begin{pmatrix}\hat{u}\\ \hat{v}\\ \hat{w}\end{pmatrix}=\exp\left(\alpha\frac{\partial}{\partial\phi}\right)\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{u}\\ \hat{v}\\ \hat{w}\end{pmatrix}. \quad (3.3)$$

This straightforward geometric construction needs to be modified in order to incorporate spin angular momentum, as discussed later.

Lie Algebra

We can derive the Lie Algebra between the three position and three momentum operators of E3 to obtain a total of 21 commutation relations (Willard Miller, 1964):

$$\begin{aligned}&\left[\hat{u},\hat{J}_v\right]=i\varepsilon_{uvw}\hat{w} && (9\ \textit{relations}),\\ &\left[\hat{J}_u,\hat{J}_v\right]=i\varepsilon_{uvw}\hat{J}_w && (6\ \textit{relations}),\\ &[\hat{u},\hat{u}]=[\hat{u},\hat{v}]=[\hat{v},\hat{w}]=0 && (6\ \textit{relations}).\end{aligned} \quad (3.4)$$

The position operators form an Abelian ideal. The Lie Algebra of E3 has two quadratic Casimir operators given by $\hat{C}=\hat{u}^2+\hat{v}^2+\hat{w}^2$ and by $\hat{O}=\hat{u}\hat{J}_u+\hat{v}\hat{J}_v+\hat{w}\hat{J}_w$, which commute with all operators. The first, $\hat{C}=1$ represents the unit radius of the $\mathbf{S^2}$ sphere and the second $\hat{O}=0$ captures the spin zero condition implicit in the angular momentum operator representation used in (3.2).

We can introduce a total momentum operator by defining $\hat{J}^2\equiv\hat{J}_u^2+\hat{J}_v^2+\hat{J}_w^2$ and find that this commutes with all the components of momentum, but not with any of the position operators:

$$\begin{aligned}&\left[\hat{J}_u,\hat{J}^2\right]=\left[\hat{J}_v,\hat{J}^2\right]=\left[\hat{J}_w,\hat{J}^2\right]=0,\\&\left[\hat{J}^2,\hat{w}\right]=i\left\{\hat{u},\hat{J}_v\right\}-i\left\{\hat{J}_u,\hat{v}\right\}\quad \textit{and cyclic permutations}.\end{aligned}\tag{3.5}$$

We can thus characterise the eigenstates of an E3 system in terms of the two eigenvalues of a Cartan sub algebra (or maximal commuting subset) of two operators: $\hat{J}^2$ plus a chosen component of angular momentum, say $\hat{J}_w$. The Lie Algebra can then be recast in terms of $\hat{J}_w$, $\hat{w}$ and the raising and lowering operators $\hat{J}_\pm=\hat{J}_u\pm i\hat{J}_v$ and $\hat{W}_\pm=\hat{u}\pm i\hat{v}$. This leads to the restated commutation relations:

$$\left.\begin{aligned}&\left[\hat{J}_w,\hat{J}_\pm\right]=\pm\hat{J}_\pm,\\&\left[\hat{J}_+,\hat{J}_-\right]=2\hat{J}_w,\\&\left[\hat{w},\hat{J}_\pm\right]=\left[\hat{J}_w,\hat{W}_\pm\right]=\pm\hat{W}_\pm,\\&\left[\hat{W}_+,\hat{J}_-\right]=\left[\hat{J}_+,\hat{W}_-\right]=2\hat{w},\\&\left[\hat{w},\hat{W}_\pm\right]=\left[\hat{W}_\pm,\hat{W}_\pm\right]=\left[\hat{J}_w,\hat{w}\right]=\left[\hat{W}_+,\hat{J}_+\right]=\left[\hat{W}_-,\hat{J}_-\right]=\\&\qquad\left[\hat{J}_w,\hat{J}_w\right]=\left[\hat{J}_+,\hat{J}_+\right]=\left[\hat{J}_-,\hat{J}_-\right]=0,\end{aligned}\right\}\ 21\text{ relations}\tag{3.6}$$

$$\begin{aligned}&\left[\hat{J}^2,\hat{J}_\pm\right]=0,\\&\left[\hat{J}^2,\hat{W}_\pm\right]=\pm\left\{\hat{J}_w,\hat{W}_\pm\right\}\mp\left\{\hat{w},\hat{J}_\pm\right\}.\end{aligned}\tag{3.7}$$

Both the $\hat{J}_\pm$ and the $\hat{W}_\pm$ raising and lowering operators increment or decrement the $\hat{J}_w$ eigenstate by an integer eigenvalue step, but only the $\hat{J}_\pm$ operators commute with $\hat{J}^2$. Thus the $\hat{J}_\pm$ operators raise/lower momentum eigenstates within a single SO(3) (or SU(2)) irrep without affecting the total angular momentum, while the $\hat{W}_\pm$ operators generally mix up the irreps.

In the geometric representation defined by (3.1) and (3.2), the raising and lowering operators can be expressed as:

$$\hat{J}_{\pm} = \hat{J}_u \pm i\hat{J}_v = e^{\pm i\phi}\left(i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right), \qquad (3.8)$$
$$\hat{W}_{\pm} = \hat{u} \pm i\hat{v} = \sin\theta e^{\pm i\phi},$$

and the total angular momentum operator takes the form of the spherical Laplacian operator:

$$\hat{J}^2 = \hat{J}_u^2 + \hat{J}_v^2 + \hat{J}_w^2 = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \equiv -\nabla_{\theta\phi}^2. \qquad (3.9)$$

The $\hat{J}_w$, $\hat{J}^2$ and position operators are Hermitian while the $\hat{J}_{\pm}$ and $\hat{W}_{\pm}$ operators are PT symmetric under Parity chosen as $\mathcal{P}:(\hat{u},\hat{v},\hat{w}) \to (\hat{u},-\hat{v},\hat{w})$ and $\phi \to -\phi$.

Basis for Eigenstates

The Hamiltonian and associated Schrodinger equation for the free spin zero E3 system can be written in terms of the total angular momentum operator as:

$$\hat{H}\psi = \hat{J}^2\psi = -\nabla_{\theta\phi}^2\psi = E\psi, \qquad (3.10)$$

where we omit constants of proportionality relating to mass, radius of S^2, etc, for simplicity of presentation. The eigenstates are thus provided by the spherical harmonics $Y_{l,m}(\theta,\phi)$, which are complex valued functions on the S^2 sphere (Davies and Betts, 1994):

$$\hat{J}^2 Y_{l,m}(\theta,\phi) = -\nabla_{\theta\phi}^2 Y_{l,m}(\theta,\phi) = l(l+1)Y_{l,m}(\theta,\phi). \qquad (3.11)$$

The spherical harmonics provide a complete set of basis functions in terms of which a general function on S^2 can be expanded. The energy eigenvalues E are degenerate with respect to m and are given by the series *l(l+1)*, where *l* is an integer and $-l \le m \le l$.

These solutions can, however, only be used to examine perturbations of bosonic spin zero wavefunctions, and we wish to study general spin representations.

Spin Representations

The description of SO(3) rotations requires the use of three angular variables or Euler angles to capture the three SO(3) degrees of freedom and therefore the above description of states using the two orbital angles (θ,ϕ) is incomplete. The orbital angular momentum operators (3.2) can, however, be generalised to incorporate a third angle, by a redefinition of the angular momentum raising and lowering operators (Brink and Satchler, 1968):

$$\hat{J}_{\pm} \to \hat{J}_{\pm} + \hat{S}_{\pm}, \text{ where } \hat{S}_{\pm} = -i\frac{\hat{W}_{\pm}}{\hat{W}_{+}\hat{W}_{-}}\frac{\partial}{\partial\chi} = -i\frac{e^{\pm i\phi}}{\sin\theta}\frac{\partial}{\partial\chi}. \tag{3.12}$$

The three Euler angles (θ,ϕ,χ) fully characterise the possible rotations on the manifold. It can readily be verified that these new angular momentum operators obey the same Lie Algebra relations (3.6), due to the particular construction of the $\hat{S}_{\pm}$ terms.

The harmonic eigenstate solutions to equations incorporating the generalised operators are of the form:

$$Y^{l}_{s,m}(\theta,\phi,\chi) = Y^{l}_{s,m}(\theta,\phi)\exp(is\chi) \tag{3.13}$$

Combining (3.13) with (3.12) leads to an equivalent form of the generalised raising and lowering operators (Willard Miller, 1964) that incorporates a new quantum number s, where s can be positive or negative and is an integer for periodic representations or a half-integer for antiperiodic representations:

$$\begin{aligned}
\hat{J}_u &= i\left(\sin\phi\frac{\partial}{\partial\theta}+\frac{\cos\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)+s\frac{\cos\phi}{\sin\theta},\\
\hat{J}_v &= i\left(-\cos\phi\frac{\partial}{\partial\theta}+\frac{\sin\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)+s\frac{\sin\phi}{\sin\theta},\\
\hat{J}_w &= -i\frac{\partial}{\partial\phi},\\
\hat{J}_\pm &= e^{\pm i\phi}\left(i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\pm\frac{\partial}{\partial\theta}+\frac{s}{\sin\theta}\right).
\end{aligned} \tag{3.14}$$

This redefinition leaves the Lie Algebra commutation relations (3.4) unaltered, but changes the value of the Casimir operator to $\hat{O}=s$, measuring the spin of the representation. The total angular momentum operator also acquires extra spin terms:

$$\hat{J}^2=\hat{J}_u^2+\hat{J}_v^2+\hat{J}_w^2=-\nabla_{\theta\phi}^2+2is\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi}+\frac{s^2}{\sin^2\theta}. \tag{3.15}$$

Algebraically, the modified operators $\hat{J}_u$, $\hat{J}_v$ and $\hat{J}_w$ now include components $\left(\ s\hat{u}/\left(\hat{u}^2+\hat{v}^2\right),\ \ s\hat{v}/\left(\hat{u}^2+\hat{v}^2\right),\ \ 0\ \right)$, which represent an outward pointing vector field parallel to the equatorial plane, and the operators exponentiate to generate U(1) phase shifts in addition to rotations over θ and ϕ. This twisting feature of representations is sometimes termed a *magnetic monopole* (Tsiganov, 2006), (Isham, 1983). For brevity we refer to it simply as *spin*.

The SO(3) sub algebra of E3 is isomorphic to SU(2), which gives a double cover of SO(3). We can therefore think of the E3 algebra either as acting on a configuration space of $S^2\otimes SO(3)$, or on $S^2\otimes SU(2)$. The half-integer spin representations correspond to the latter case.

Each pair of values for s and *l* defines the top state of an SO(3)/SU(2) irrep. The top state has angular momentum component m=*l* and its angular representation is

determined by requiring that it is annihilated by the $\hat{J}_+$ raising operator. A general state can then be found by applying the $\hat{J}_-$ lowering operator.

The angular representation, excluding normalisation, of one of these generalised spherical harmonics is given (Willard Miller, 1964) by:

$$Y^l_{s,m}(\theta,\phi,\chi) \propto (1-\cos\theta)^{\frac{s-m}{2}}(1+\cos\theta)^{-\frac{s+m}{2}}\frac{d^{l-m}}{d(\cos\theta)^{l-m}}\left[(1-\cos\theta)^{l-s}(1+\cos\theta)^{l+s}\right]e^{im\phi}e^{is\chi}. \quad (3.16)$$

The principle that each irrep should close under the action of both the $\hat{J}_\pm$ raising and lowering operators leads to the requirements that l≥|m| and l≥|s| and that the quantum numbers should be either all integer or all half-integer. The integer states are 2π periodic, while the half integer states are antiperiodic.

The states constitute solutions to the Schrodinger equation:

$$\hat{J}^2 Y^l_{s,m}(\theta,\phi) = \left(-\nabla^2_{\theta\phi} + 2is\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} + \frac{s^2}{\sin^2\theta}\right)Y^l_{s,m}(\theta,\phi) = \underbrace{l(l+1)}_{E}Y^l_{s,m}(\theta,\phi). \quad (3.17)$$

Thus, each value of l defines a multiplet of degenerate states, for each value of s, labelled from m= –l to m=l, with their energy eigenvalues all given by the $l(l+1)$ series. The first few generalised spherical harmonic states for spin values of s=0, ½ and 1 are listed in **Table 1**. If s=0, the solutions reduce to the standard spherical harmonics $Y^l_m(\theta,\phi)$.

Table 1. Generalised spherical harmonics for l≤2 and s=0, s=±½ and s=±1.

l	m	s	$Y^{l}_{s,m}(\theta,\phi)$
0	0	0	1
$\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\sqrt{1\underset{ms}{\pm}\cos\theta}\ e^{\pm i\phi/2}_{m}$
1	±1	0	$\sin\theta\ e^{\pm i\phi}_{m}$
1	0	0	$\cos\theta$
1	±1	±1	$\left(1\underset{ms}{\pm}\cos\theta\right)e^{\pm i\phi}_{m}$
1	0	±1	$\sin\theta$
$\frac{3}{2}$	$\pm\frac{3}{2}$	$\pm\frac{1}{2}$	$\sin\theta\sqrt{1\underset{ms}{\pm}\cos\theta}e^{\pm i3\phi/2}_{m}$
$\frac{3}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\left(1\underset{ms}{\mp}3\cos\theta\right)\sqrt{1\underset{ms}{\pm}\cos\theta}\ e^{\pm i\phi/2}_{m}$
2	±2	0	$\sin^2\theta e^{\pm i2\phi}_{m}$
2	±1	0	$\sin\theta\cos\theta e^{\pm i\phi}_{m}$
2	0	0	$\left(1-3\cos^2\theta\right)$
2	±2	±1	$\left(1\underset{ms}{\pm}\cos\theta\right)\sin\theta e^{\pm i2\phi}_{m}$
2	±1	±1	$\left(1\underset{ms}{\mp}2\cos\theta\right)\left(1\underset{ms}{\pm}\cos\theta\right)e^{\pm i\phi}_{m}$
2	0	±1	$\sin\theta\cos\theta$

It can be seen that the $\hat{W}_{\pm}$ operators link top or bottom states respectively in adjacent irreps with the same spin value. Also, Parity and Time Reversal transformations switch between the m-states and s-states within each *l*-multiplet. If we consider two possible choices for PT:

$$\begin{aligned} \mathcal{PT}:&\ (\theta,\phi,i,s)\to(\theta,-\phi,-i,s) \\ &\text{or} \\ \mathcal{PT}:&\ (\theta,\phi,i,s)\to(\pi-\theta,\phi,-i,s). \end{aligned} \tag{3.18}$$

we find the former leaves all the states invariant, while the second reverses the sign of m. In all cases, however, the irreps as a whole are invariant under PT.

These states can be normalised to provide a complete orthonormal basis (Willard Miller, 1964) and we can use this basis to analyse the general spin eigenstates of E3 Hamiltonians with PT symmetric potentials.

Hermitian and PT Symmetric Hamiltonians

Let us suppose a potential $\hat{V}(\hat{u},\hat{v},\hat{w},\ldots)$ is applied to an E3 system with quantum numbers (l,m,s). In order to find the new eigenvalues, we can work in a matrix formalism using the complete free eigenstate basis and seek the solutions to the equation:

$$\sum_{l,m,s} \underbrace{\left\langle Y^{l'}_{s',m'} \middle| l(l+1) + \hat{V}(\hat{u},\hat{v},\hat{w},\ldots) \middle| Y^{l}_{s,m} \right\rangle}_{Hamiltonian} a_{l,m,s} = \lambda(\lambda+1)\, a_{l',m',s'}. \qquad (3.19)$$

In principle, to achieve completeness we have to diagonalise over all quantum numbers, however, when the potential has symmetry, we can work with a subset. We need to evaluate Hamiltonian matrix elements for a range of potentials constructed out of the Lie Algebra and there are established recursion relations that assist in this (Willard Miller, 1964):

$$\hat{w}\left|\mathrm{Y}^{l}_{s,m}\right\rangle = \begin{bmatrix} \sqrt{\dfrac{(l+m+1)(l-m+1)(l+s+1)(l-s+1)}{(l+1)^2(2l+3)(2l+1)}}\left|\mathrm{Y}^{l+1}_{s,m}\right\rangle + \ldots \\ +\dfrac{sm}{l(l+1)}\left|\mathrm{Y}^{l}_{s,m}\right\rangle + \sqrt{\dfrac{(l+m)(l-m)(l+s)(l-s)}{l^2(2l+1)(2l-1)}}\left|\mathrm{Y}^{l-1}_{s,m}\right\rangle \end{bmatrix}, \qquad (3.20)$$

$$W_{\pm}\left|Y^{l}_{s,m}\right\rangle = \begin{bmatrix} \mp\sqrt{\dfrac{(l+s+1)(l-s+1)(l\pm m+1)(l\pm m+2)}{(l+1)^2(2l+3)(2l+1)}}\left|Y^{l+1}_{s,m\pm1}\right\rangle + \ldots \\ +\sqrt{(l\pm m+1)(l\mp m)}\,\dfrac{s}{l(l+1)}\left|Y^{l}_{s,m\pm1}\right\rangle + \ldots \\ \pm\sqrt{\dfrac{(l+s)(l-s)(l\mp m-1)(l\mp m)}{l^2(2l+1)(2l-1)}}\left|Y^{l-1}_{s,m\pm1}\right\rangle \end{bmatrix}. \qquad (3.21)$$

We thus have the tools to analyse the impact of a perturbing potential on the eigenvalues. We shall consider in detail the cases of potentials parameterised by g:

$$\begin{aligned}\hat{V} &= g\hat{w} = g\,\cos\theta,\\ \hat{V} &= g\hat{u} = g\,\sin\theta\cos\phi = \frac{g}{2}\left(W_{+} + W_{-}\right).\end{aligned} \tag{3.22}$$

For g=0 the energy eigenvalues are given by the series $l(l+1)$. For g real, both these Hamiltonians are Hermitian, and for g imaginary, both are PT-symmetric:

$$\begin{aligned}\mathcal{PT}:\quad & (\hat{u},\hat{v},\hat{w},i) && \to (\hat{u},\hat{v},-\hat{w},-i),\\ & i\ \mathrm{g}\hat{w} && \to i\ \mathrm{g}\hat{w}\\ & && \text{or}\\ \mathcal{PT}:\quad & (\hat{u},\hat{v},\hat{w},i) && \to (-\hat{u},\hat{v},\hat{w},-i),\\ & i\ \mathrm{g}\hat{u} && \to i\ \mathrm{g}\hat{u}.\end{aligned} \tag{3.23}$$

Consider first the directional potential $\hat{V} = g\hat{w}$. Since the potential is invariant under rotation by φ, the m eigenvalues are unaffected. However, the potential causes mixing between l-states which results in changes in the $l(l+1)$ energy eigenvalue, which we denote as λ(λ+1). The matrix element is calculated by substituting the recursion relation (3.20) into (3.22) and (3.19) to obtain the matrix Hamiltonian:

$$\mathcal{H} = +l(l+1)\delta^{l',l} + g\left[\begin{array}{l}\sqrt{\dfrac{(l+m+1)(l-m+1)(l+s+1)(l-s+1)}{(l+1)^2(2l+3)(2l+1)}}\,\delta^{l',l+1} + \ldots\\ +\dfrac{sm}{l(l+1)}\,\delta^{l',l} + \sqrt{\dfrac{(l+m)(l-m)(l+s)(l-s)}{l^2(2l+1)(2l-1)}}\,\delta^{l',l-1}\end{array}\right]. \tag{3.24}$$

Assuming l≥|m|≥|s| and fixing the minimum value of l by n=|m|, the symmetric matrix over l′l that we need to diagonalise takes the form:

$\mathcal{H}$	$\lvert n,m,s\rangle$	$\lvert n+1,m,s\rangle$	…	$\lvert n+k,m,s\rangle$	…
$\langle n,m,s\rvert$	$n(n+1)+g\frac{sm}{n(n+1)}$	$g\sqrt{\ldots}$	0	0	0
$\langle n+1,m,s\rvert$	$g\sqrt{\frac{(n+1+s)(n+1-s)}{(n+1)^2(2n+3)}}$	…	…	0	0
…	0	…	…	…	0
$\langle n+k,m,s\rvert$	0	0	…	$(n+k)(n+k+1)+g\frac{sm}{(n+k)(n+k+1)}$	$g\sqrt{\ldots}$
…	0	0	0	$g\sqrt{\frac{(2n+k+1)(k+1)(n+k+1+s)(n+k+1-s)}{(2n+2k+1)(2n+2k+3)(n+k+1)^2}}$	…

(3.25)

This *symmetric tridiagonal* form does not generally arise, but is a feature peculiar to the g cosθ potential. The diagonal terms scale as the square of the row/column index k as the matrix increases in size, while the off-diagonal terms tend to g/2, so just as in the case of E2, we observe relatively little mixing between widely separated *l*-states. Also, we can note the presence on the diagonal of the parameter g. Thus, whenever g is imaginary, any set of *l*-states for non-zero spin s≠0 and m≠0 will have a trace that is imaginary, entailing at least one imaginary eigenvalue.

Consider now the directional potential $V = g\hat{u}$. This is related to the $V = g\hat{w}$ potential by a simple rotation and therefore we expect to find the same results for zero spin. However, if spin is non-zero, we find different behaviours due to the different orientation relative to the spin direction. The matrix element is calculated by substituting the recursion relation (3.21) into (3.22) and (3.19) to obtain the matrix Hamiltonian:

$$\mathcal{H} = l(l+1)\delta_{l',l}\delta_{m',m} + \frac{g}{2}\left[\begin{array}{l} -\sqrt{\frac{(l+s+1)(l-s+1)(l+m+1)(l+m+2)}{(l+1)^2(2l+3)(2l+1)}}\delta_{l',l+1}\delta_{m',m+1}\cdots \\ +\sqrt{\frac{(l+s+1)(l-s+1)(l-m+1)(l-m+2)}{(l+1)^2(2l+3)(2l+1)}}\delta_{l',l+1}\cdots \\ +\sqrt{(l+m+1)(l-m)}\,\frac{s}{l(l+1)}\delta_{l',l}\delta_{m',m+1}\cdots \\ +\sqrt{(l-m+1)(l+m)}\,\frac{s}{l(l+1)}\delta_{l',l}\delta_{m',m-1}\cdots \\ +\sqrt{\frac{(l+s)(l-s)(l-m-1)(l-m)}{l^2(2l+1)(2l-1)}}\delta_{l',l-1}\delta_{m',m+1}\cdots \\ -\sqrt{\frac{(l+s)(l-s)(l+m-1)(l+m)}{l^2(2l+1)(2l-1)}}\delta_{l',l-1}\delta_{m',m-1} \end{array}\right]. \tag{3.26}$$

This potential mixes up adjacent m states in the same and neighbouring l-multiplets with the same s value. In the zero spin case, the mixing is limited to adjacent l-multiplets. When constructing the mixing matrix, it is necessary to fix a spin value and then cycle through all the possible m states taking each l-multiplet in turn. This leads to a *symmetric block tridiagonal structure* for the Hamiltonian, as illustrated for s=0 and *l* ranging from 0 to 2:

$\lvert l,m,s\rangle$	$\lvert 0,0,0\rangle$	$\lvert 1,-1,0\rangle$	$\lvert 1,0,0\rangle$	$\lvert 1,1,0\rangle$	$\lvert 2,-2,0\rangle$	$\lvert 2,-1,0\rangle$	$\lvert 2,0,0\rangle$	$\lvert 2,1,0\rangle$	$\lvert 2,2,0\rangle$
$\langle 0,0,0\rvert$	0	$\frac{g}{\sqrt{6}}$	0	$-\frac{g}{\sqrt{6}}$	0	0	0	0	0
$\langle 1,-1,0\rvert$	$\frac{g}{\sqrt{6}}$	2	0	0	$\frac{g}{\sqrt{5}}$	0	$-\frac{g}{\sqrt{30}}$	0	0
$\langle 1,0,0\rvert$	0	0	2	0	0	$\frac{g}{\sqrt{10}}$	0	$-\frac{g}{\sqrt{10}}$	0
$\langle 1,1,0\rvert$	$-\frac{g}{\sqrt{6}}$	0	0	2	0	0	$\frac{g}{\sqrt{30}}$	0	$-\frac{g}{\sqrt{5}}$
$\langle 2,-2,0\rvert$	0	$\frac{g}{\sqrt{5}}$	0	0	6	0	0	0	0
$\langle 2,-1,0\rvert$	0	0	$\frac{g}{\sqrt{10}}$	0	0	6	0	0	0
$\langle 2,0,0\rvert$	0	$-\frac{g}{\sqrt{30}}$	0	$\frac{g}{\sqrt{30}}$	0	0	6	0	0
$\langle 2,1,0\rvert$	0	0	$-\frac{g}{\sqrt{10}}$	0	0	0	0	6	0
$\langle 2,2,0\rvert$	0	0	0	$-\frac{g}{\sqrt{5}}$	0	0	0	0	6

(3.27)

The parameter g does not appear on the diagonal for any states and this meets a necessary condition for unbroken PT symmetry to arise when g is imaginary. Whether the block tridiagonal form is sufficient for unbroken PT symmetry to arise, however, needs to be explored numerically. (A matrix of approximately 100 by 100, covering the first 10 *l*-multiplets was diagonalised to generate the figures that follow).

Eigenvalue Splitting Patterns: $g\hat{w}$ or $g\cos\theta$ potential

Figure 7, Figure 8 and Figure 9 show the distortions of the first six or so energy eigenvalues under a Hermitian real potential $V(\theta)=g\cos\theta$ applied to states associated with component angular momentum numbers *m* up to 2 and spin values of zero (bosonic), spin ½ (fermionic) and spin 1 (bosonic). Although the eigenvalues become negative for large real potentials, they remain real in all cases. The potential breaks the degeneracy between the various m-states and the eigenvalues split. The bosonic spin zero energy perturbation is symmetric between positive and negative values of the potential, but the fermionic perturbations are skewed reflecting the sign of spin chosen relative to the sign of angular momentum component m. The splitting pattern of spin one bosonic energy levels is similar to that of spin zero bosons if the component angular momentum m=0, or otherwise similar to fermions in the case that m is non-zero.

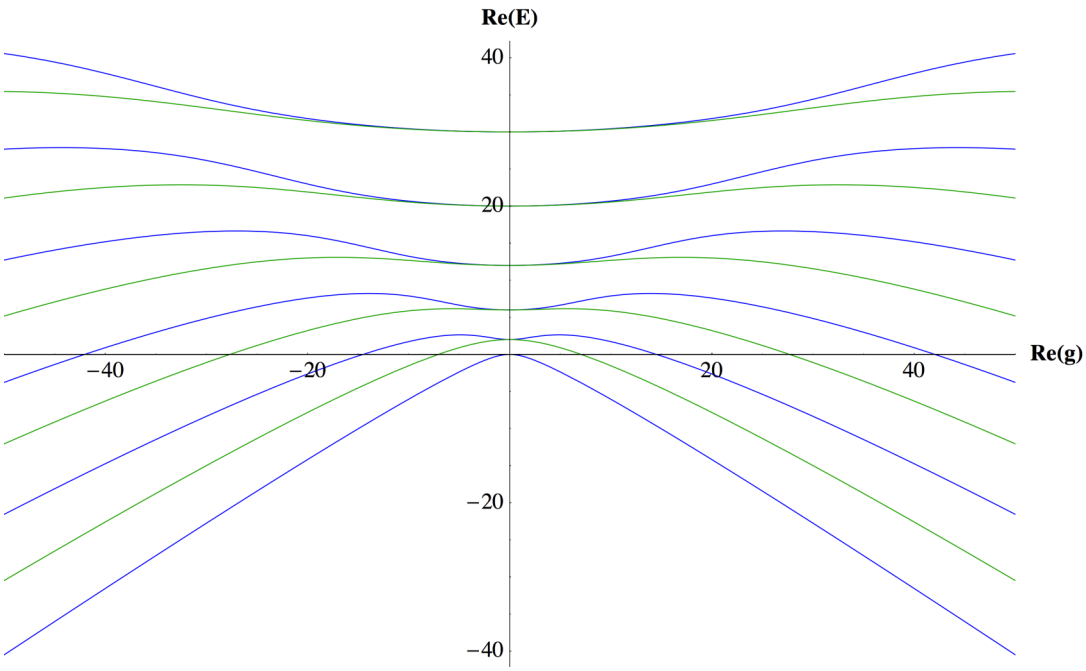

Figure 7. Eigenvalues for E3 bosonic (s=0) states under a real g cosθ potential. The first six m=0 eigenvalues (blue) and the first five m=1 eigenvalues (green) are shown. Intercepts on the E axis are given by the l(l+1) series for l=0 to 5.

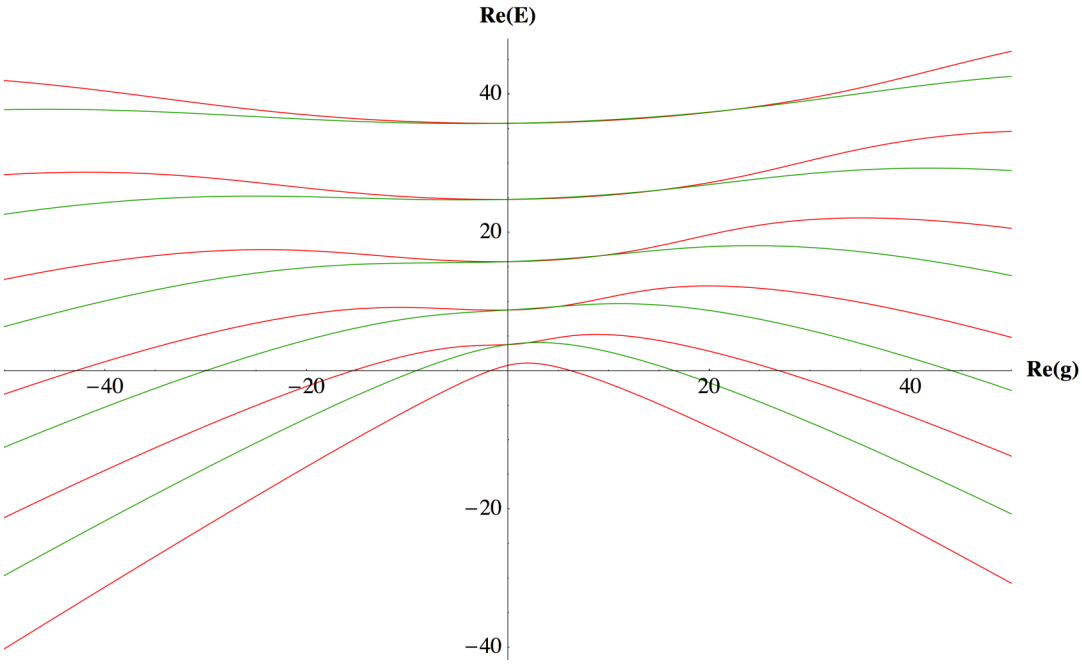

Figure 8. Eigenvalues for the E3 fermionic (s=1/2) states under a real g cosθ potential. The first six m=1/2 eigenvalues (red) and the first five m=3/2 eigenvalues (green) are shown. Intercepts on the E axis are given by the l(l+1) series for l=1/2 to 11/2.

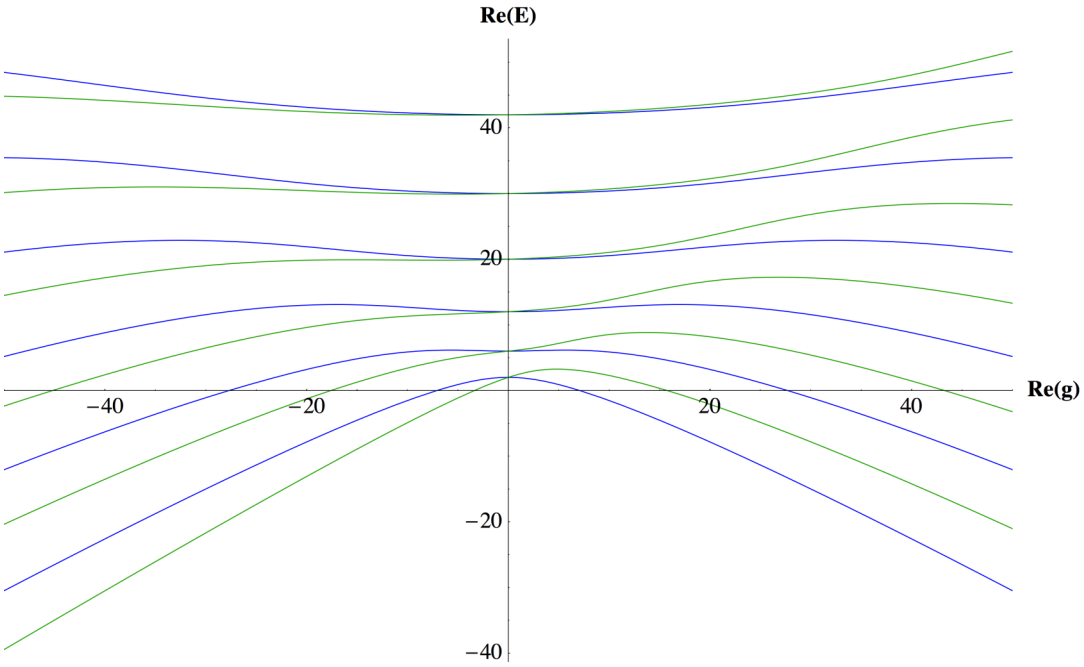

Figure 9 Eigenvalues for E3 bosonic (s=1) states under a real g cosθ potential. The first six m=0 eigenvalues (blue) and the first six m=1 eigenvalues (green) are shown. Intercepts on the E axis are given by the l(l+1) series for l=1 to 6.

The eigenvalues associated with a PT symmetric imaginary g cosθ potential are shown in Figure 10 through Figure 15 for spin representations of zero, ½ and one. The eigenvalue splitting and mixing patterns differ significantly between the spin representations. The bosonic spin zero eigenvalues exhibit regions of PT-symmetry for small magnitudes of the potential, however, in the case of fermions, the eigenvalues become complex for an arbitrarily small PT symmetric potential and there is no region of unbroken PT symmetry. The spin one boson energy levels mix like those of spin zero bosons only if they have an angular momentum component m=0; otherwise they behave like fermions.

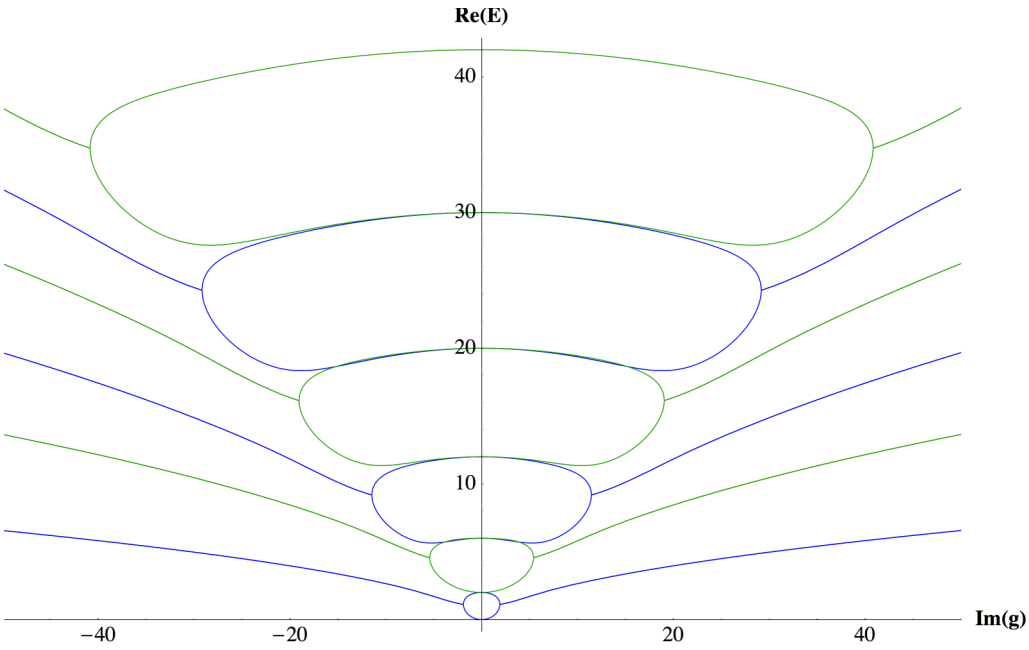

Figure 10. Real components of eigenvalues for E3 bosonic (s=0) states under an imaginary g cosθ potential. The first six m=0 eigenvalues (blue) and the first six m=1 eigenvalues (green) are shown. Intercepts on the E axis are given by l(l+1) for l=0 to 6.

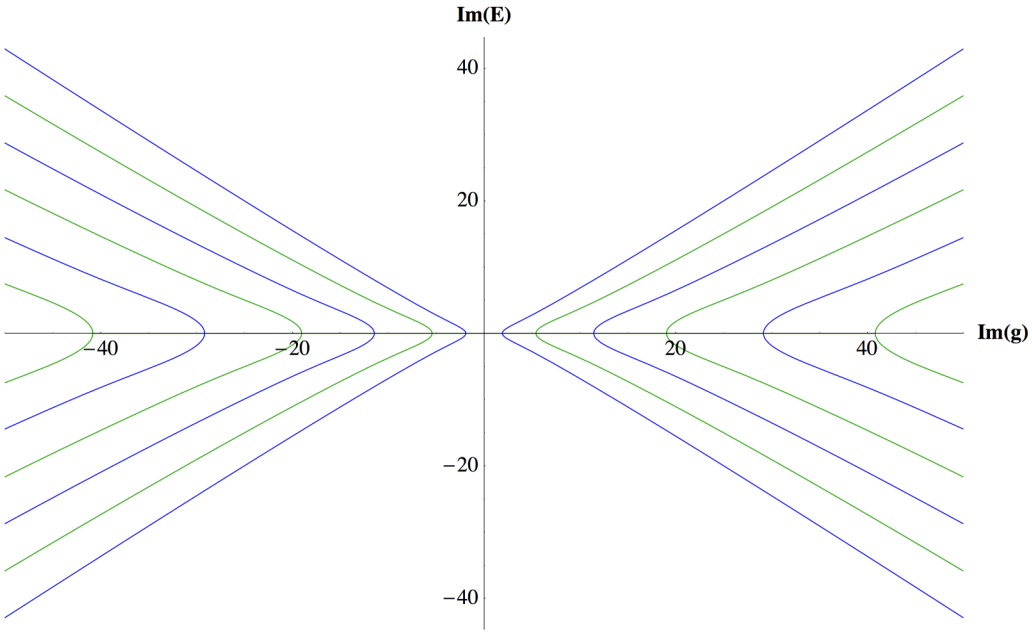

Figure 11. Imaginary components of eigenvalues for E3 bosonic (s=0) states under an imaginary g cosθ potential. The first six m=0 eigenvalues (blue) and the first six m=1 eigenvalues (green) are shown. *There is a region of unbroken PT symmetry.*

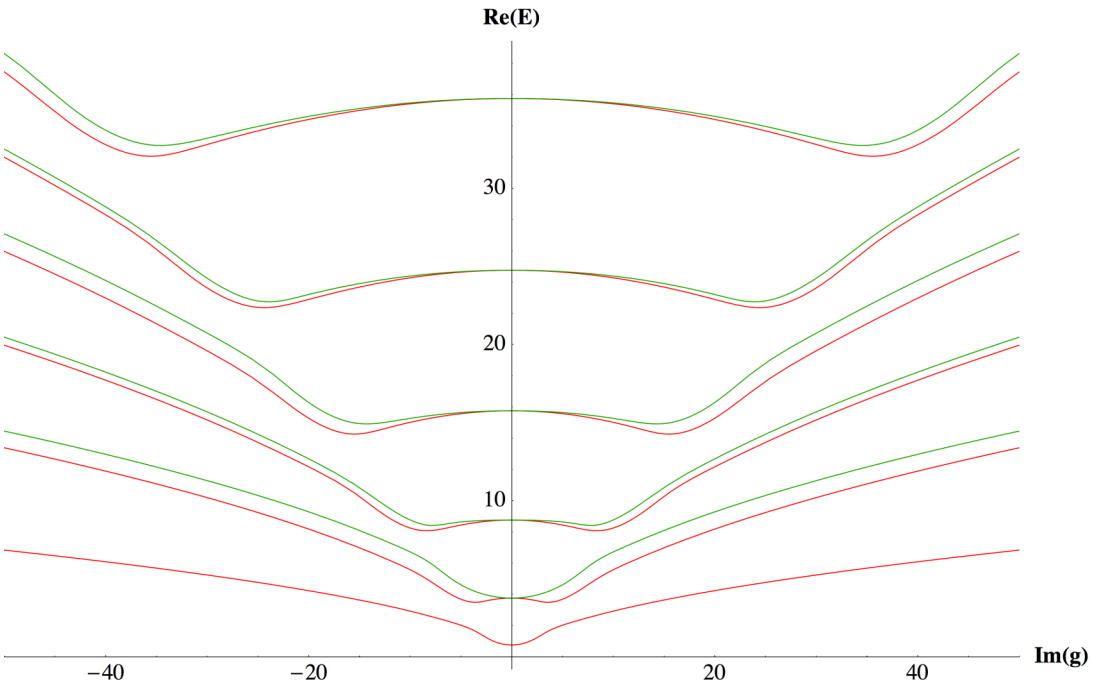

Figure 12. Real components of eigenvalues for E3 fermionic (s=1/2) states under an imaginary g cosθ potential. The first six m=1/2 eigenvalues (red) and the first five m=3/2 eigenvalues (green) are shown. Intercepts on the E axis are given by l(l+1) for l=1/2 to 11/2.

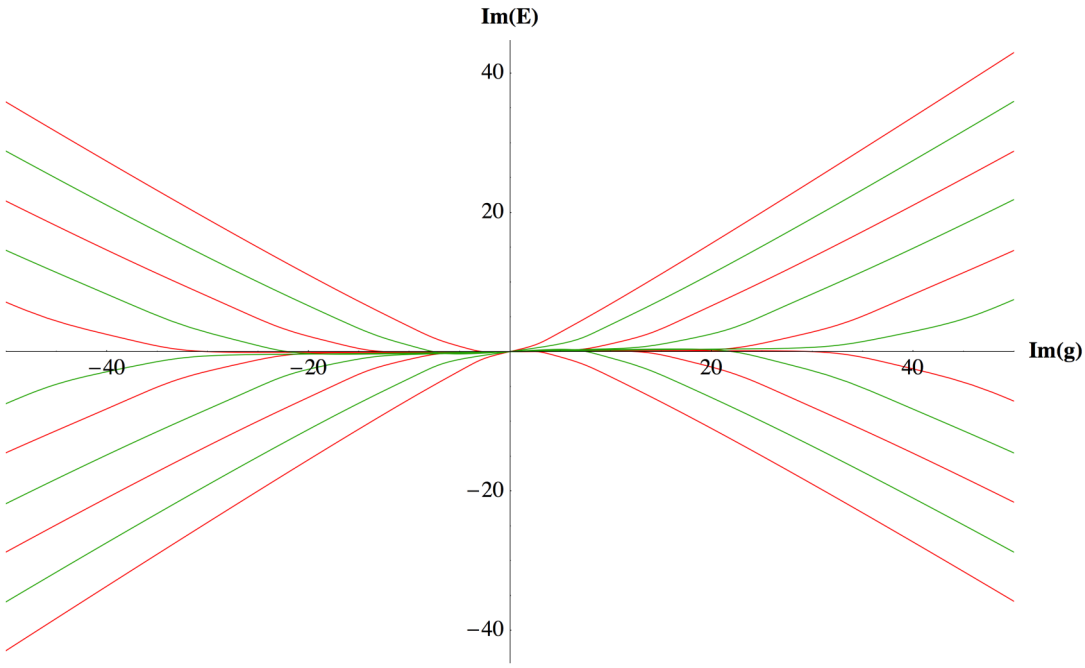

Figure 13. Imaginary components of eigenvalues for E3 fermionic (s=1/2) states under an imaginary g cosθ potential. The first six m=1/2 eigenvalues (red) and the first five m=3/2 eigenvalues (green) are shown. *There is no region of unbroken PT symmetry.*

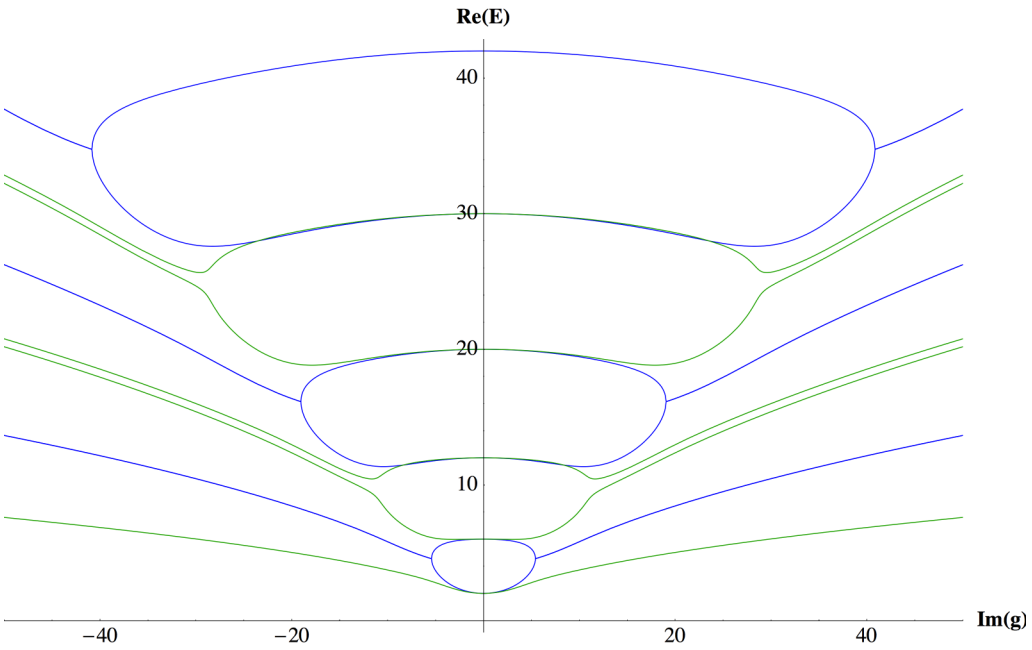

Figure 14. Real components of eigenvalues for E3 bosonic (s=1) states under an imaginary g cosθ potential. The first six m=0 eigenvalues (blue) and the first five m=1 eigenvalues (green) are shown. Intercepts on the E axis are given by l(l+1) for l=1 to 6.

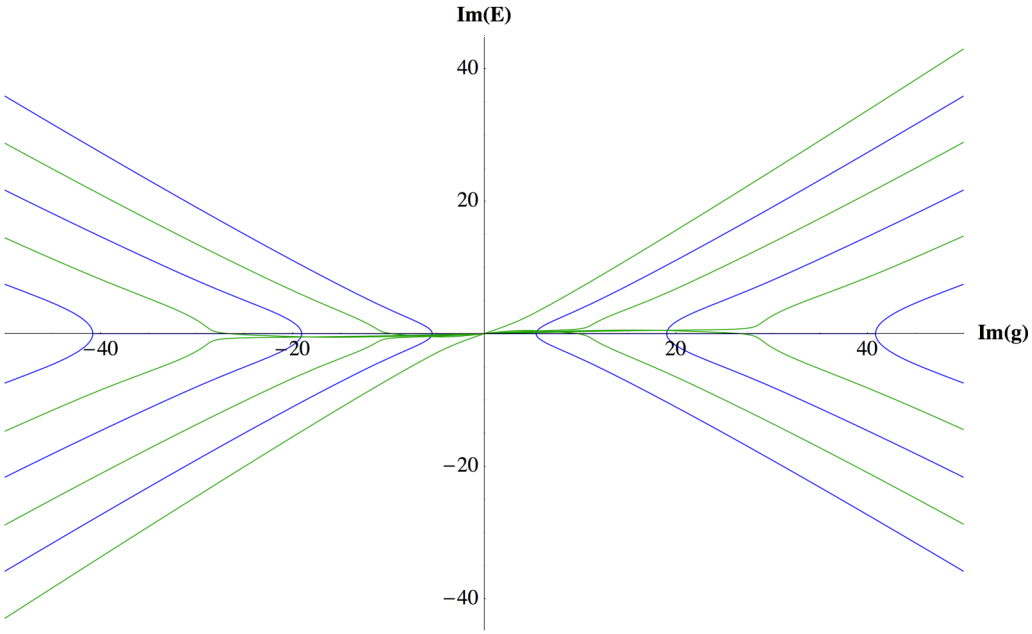

Figure 15. Imaginary components of eigenvalues for E3 bosonic (s=1) states under an imaginary g cosθ potential. The first six m=0 eigenvalues (blue) and the first five m=1 eigenvalues (green) are shown. *There is a region of unbroken PT symmetry only for m=0 states.*

The spin zero bosons follow the E2 pattern, with pairs of eigenstates, starting with the lowest pair for a given value of m, being mixed, with their eigenvalues equalising at a particular critical point for each pair. If the potential is increased beyond this critical point, the eigenvalues become complex conjugates of each other. The mixing of the first pair occurs at a critical value of Im(g)= 1.8995 for the m=0 bosonic series. The critical values at which PT symmetry is broken for the m=1 series and m=2 series are increased to Im(g)= 5.4137 and Im(g)= 10.4289 (and correspondingly for higher values of angular momentum component m).

Eigenstate Pairs $\{l_1,l_2\}$	$\{m,s\}$	*Critical Value* g
$\{0,1\}$	$\{0,0\}$	1.8995 I
$\{2,3\}$	$\{0,0\}$	11.4470 I
$\{4,5\}$	$\{0,0\}$	29.1571 I
$\{1,2\}$	$\{1,0\}or\{0,1\}$	5.4137 I
$\{3,4\}$	$\{1,0\}or\{0,1\}$	19.0367 I
$\{5,6\}$	$\{1,0\}or\{0,1\}$	40.8288 I
$\{2,3\}$	$\{2,0\}or\{0,2\}$	10.4289 I
$\{4,5\}$	$\{2,0\}or\{0,2\}$	28.1583 I

(3.28)

By virtue of the invariance of the Hamiltonian matrix (3.24) under interchange of m and s, the critical points for the non-zero spin bosons with m=0 are the same as those for zero spin bosons with non-zero m-values. Below the first critical value of the potential for each {m,s} pair all the bosonic eigenvalues are real and we have a characteristic region of unbroken PT symmetry. Above these critical values for each pair, only the higher bosonic eigenvalues remain real and the PT symmetry is broken.

Eigenvalue Splitting Patterns: gû or g$\sin\theta\cos\phi$ potential

By virtue of rotational S^2 symmetry, when spin is zero, the eigenvalue splitting patterns for this $\hat{V} = g\hat{u}$ potential are related to those for $\hat{V} = g\hat{w}$ simply by the rotation of m-states. Thus, for spin zero, if g is real, the Hamiltonian based on $\hat{V} = g\hat{u}$ is Hermitian and if g is imaginary, the Hamiltonian is PT symmetric with a region of unbroken symmetry that breaks at the same critical values as $\hat{V} = g\hat{w}$.

In the case where spin is non-zero, the situation is more complicated. **Figure 16** through **Figure 19** show the impact of a $\hat{V} = g\hat{u}$ potential on fermionic (s=1/2) eigenvalues and bosonic (s=1) eigenvalues. The essential changes from the pattern for the imaginary $\hat{V} = g\hat{w}$ potential follow from the mixing of m-states, whose axis is now no longer oriented along the potential. Thus neither the fermions nor the spin 1 bosons have a region of unbroken symmetry, since the potential mixes the m=0 state with other m-states. It can also be observed that the complex splitting pattern of the m-states in the higher multiplets causes a band structure for the eigenvalues to emerge.

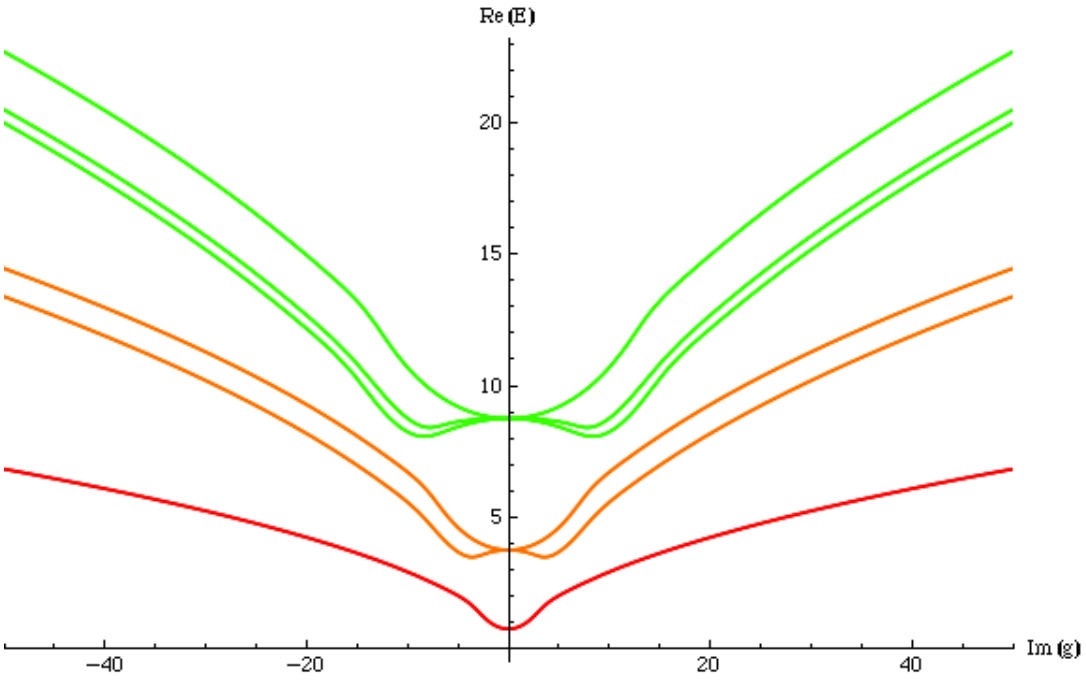

Figure 16. Real components of eigenvalues for E3 fermionic (s=1/2) states under an imaginary g sinθ cosϕ potential. The first three l-multiplets of m-states are shown, colour coded by multiplet. Intercepts on the E axis are given by l(l+1) for l=1/2 to 5/2.

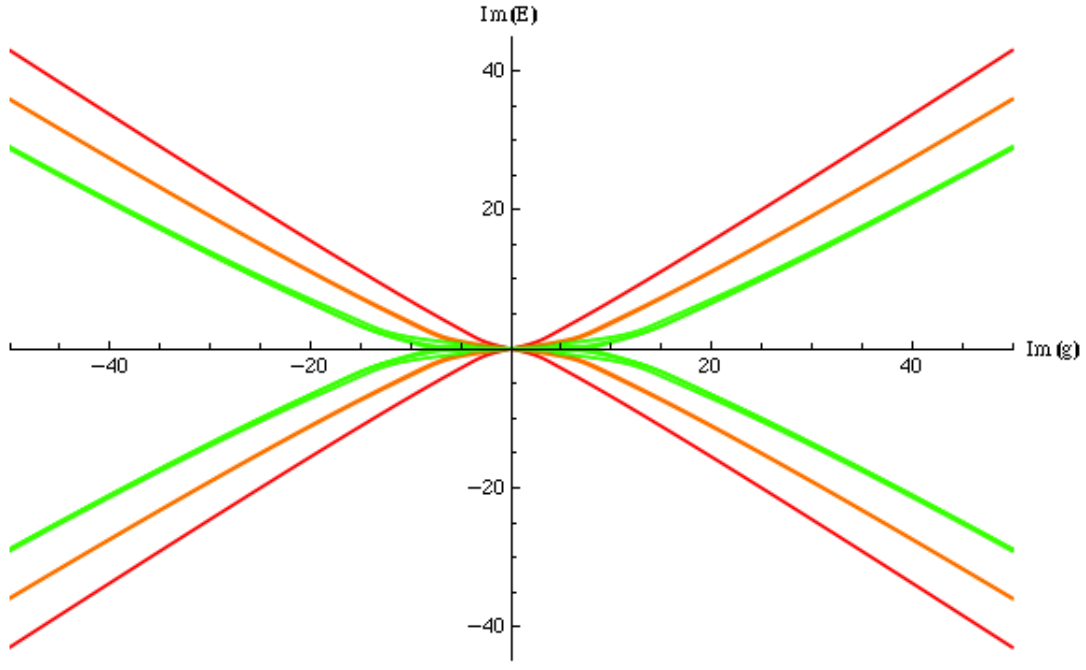

Figure 17. Imaginary components of eigenvalues for E3 fermionic (s=1/2) states under an imaginary g sinθ cosϕ potential. The first three l-multiplets are shown colour coded. *There is no region of unbroken PT symmetry.*

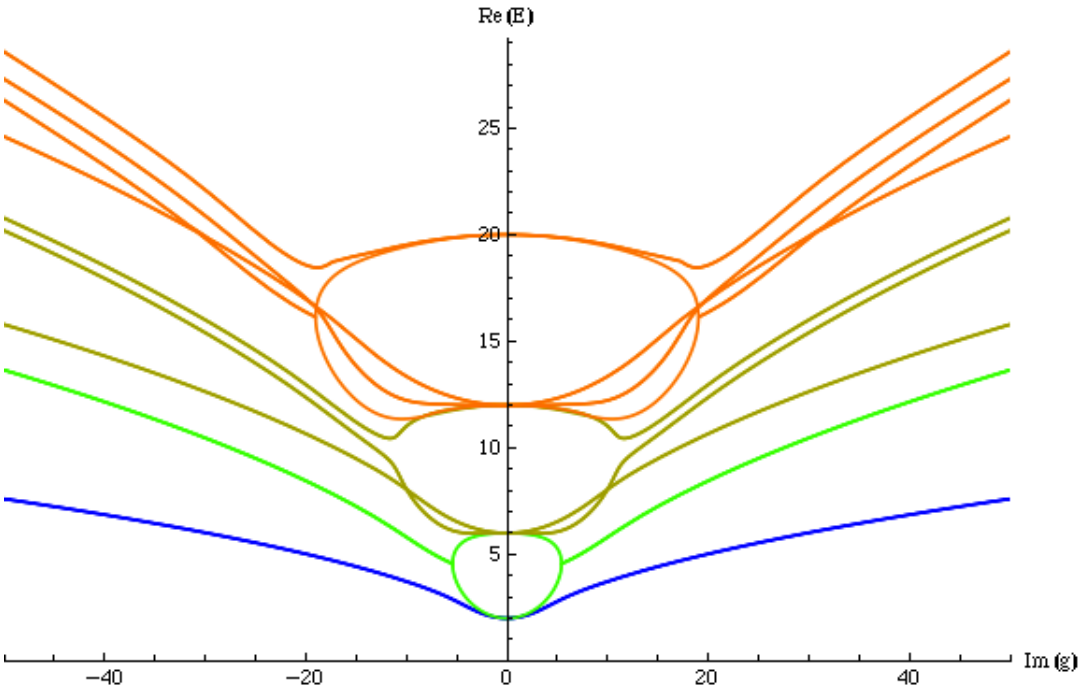

Figure 18. Real components of eigenvalues for E3 bosonic (s=1) states under an imaginary g sinθ cosϕ potential. m-states from the first four l-multiplets are shown, colour coded by pairings. Intercepts on the E axis are given by l(l+1) for l=1 to 4.

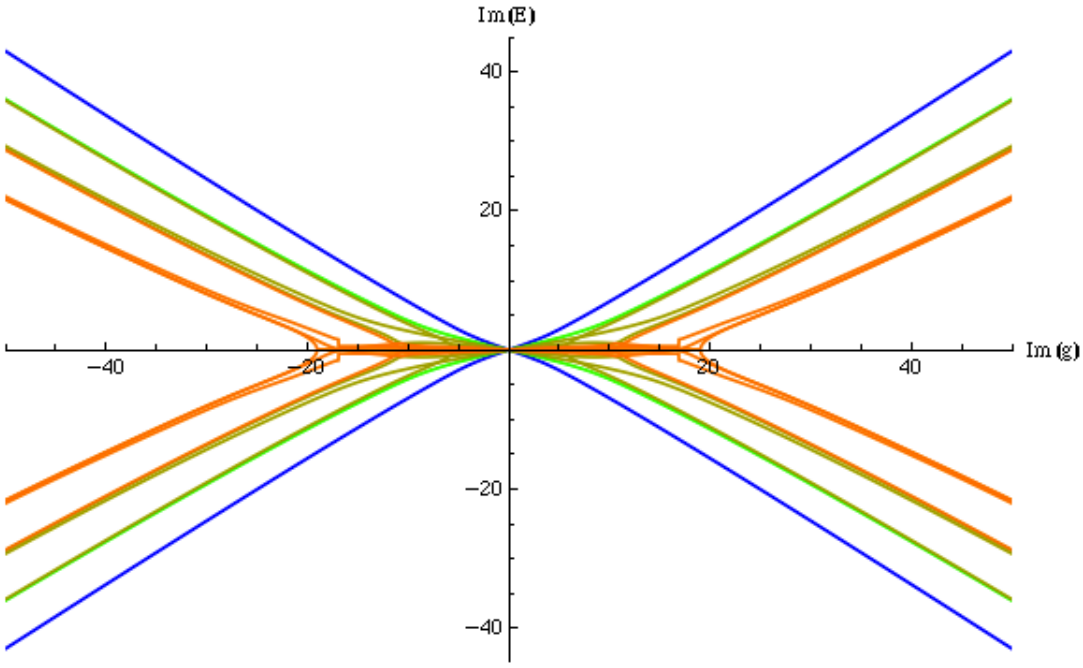

Figure 19. Imaginary components of eigenvalues for E3 bosonic (s=1) states under an imaginary g sinθ cosϕ potential. m-states from the first four l-multiplets are shown, colour coded by pairings. *There is no region of unbroken PT symmetry.*

Other potentials

We can consider briefly the anticipated behaviour of potentials built from other elements of the Lie Algebra. Thus, by S^1 symmetry, a $\hat{v}$ potential is similar to $\hat{u}$.

Having established that potentials of the form $g\hat{u}$ are Hermitian, we can also apply a coordinate similarity transformation $\mathcal{S}(\alpha)$, as in (2.12):

$$\begin{aligned} \mathcal{S}(\alpha):\ & \phi \to \phi + i\alpha, \\ & g \to \frac{g}{\cosh\alpha}, \\ & g\hat{u} \to g\left(\hat{u} - i\tanh\alpha\ \hat{v}\right). \end{aligned} \tag{3.29}$$

As in the case of E2, we find that the Hermitian position operator $\hat{u}$ transforms into the PT symmetric raising and lowering operators $W_{\pm}$ in the limiting cases as $\alpha \to \mp\infty$. These raising and lowering operators therefore have unbounded parametric regions of unbroken PT symmetry.

Also, if we construct a potential from any single raising operator $\hat{J}_{\pm}$ or $\hat{W}_{\pm}$, we obtain a mixing matrix similar to an upper triangular matrix, for which the eigenvalues are unchanged from the free series. Thus the raising and lowering operators $\hat{J}_{\pm}$ and $\hat{W}_{\pm}$ all give rise to unbounded regions of unbroken PT symmetry for both bosonic and fermionic states.

Finally we can consider a potential $V = g\hat{J}_w$. This leads to a purely diagonal Hamiltonian, which is Hermitian if g is real or PT symmetric if g is imaginary. However, this Hamiltonian is already diagonalised, with the result that if g is imaginary, then the eigenvalues are generally complex, and neither the fermionic nor the bosonic

states (with the exception of m=0 states for which the $V = g\hat{J}_w$ potential vanishes) have a region of unbroken PT symmetry.

Key Findings: E3

Quantum mechanics on the $S^2 \otimes SO(3)/SU(2)$ manifold, characterised by the E3 group, contains non-Hermitian, but PT symmetric Hamiltonians. These include the following Hamiltonians, where $\hat{J}^2$ is defined as in (3.15) to incorporate representations with non-zero spin:

$$\begin{aligned}
&\mathcal{H}(\hat{w}) = \hat{J}^2 + g\ \hat{w} && (g\ \textit{imaginary}),\\
&\mathcal{H}(\hat{u}) = \hat{J}^2 + g\ \hat{u} && (g\ \textit{imaginary}),\\
&\mathcal{H}(\hat{W}_{\pm}) = \hat{J}^2 + g\ \hat{W}_{\pm} && (g\ \textit{real or imaginary}),\\
&\mathcal{H}(\hat{J}_{\pm}) = \hat{J}^2 + g\ \hat{J}_{\pm} && (g\ \textit{real or imaginary}),\\
&\mathcal{H}(\hat{J}_w) = \hat{J}^2 + g\ \hat{J}_w && (g\ \textit{imaginary}).
\end{aligned} \tag{3.30}$$

PT symmetric Hamiltonians $\mathcal{H}(\hat{w})$ and $\mathcal{H}(\hat{u})$ have regions of unbroken PT symmetry for the bosonic zero spin zero series for imaginary g below certain critical values. Fermionic representations of E3, characterised by half-integer spin, do not possess such regions of unbroken PT symmetry. The situation of non-zero spin bosonic states is more complicated, with such states exhibiting regions of unbroken PT symmetry under $\mathcal{H}(\hat{w})$ only for m=0.

PT symmetric Hamiltonians $\mathcal{H}(W_{\pm})$ and $\mathcal{H}(J_{\pm})$, which are based on raising or lowering operators, are upper or lower triangular and have unbounded regions of unbroken PT symmetry for all spin states.

PT symmetric Hamiltonians $\mathcal{H}\left(\hat{J}_w\right)$ have regions of unbroken PT symmetry only for m=0 quantum states.

These Hamiltonians can naturally be generalised using rotational symmetries (subject to spin) and should be viewed as representatives of equivalence classes.

Chapter Four

PT Symmetry on E4

Geometric Definition

E4 is defined by the dynamics of the rotation group on the three sphere $S^3 \otimes SO(4)$. Proceeding similarly to E3, we obtain the E4 Lie Algebra by pulling back the Heisenberg commutation relations (1.1) from Cartesian coordinates on a configuration space $\mathbb{R}^4$ onto the surface of an S^3 unit hypersphere. Thus, we introduce four position operators, subject to the overall constraint $\hat{u}^2 + \hat{v}^2 + \hat{x}^2 + \hat{y}^2 = 1$ and define the six antisymmetric angular momentum operators:

$$\left.\begin{aligned}
\hat{\mathbf{J}}_{xy} &= \hat{x}p_y - \hat{y}p_x, \\
\hat{\mathbf{J}}_{vy} &= \hat{v}p_y - \hat{y}p_v, \\
\hat{\mathbf{J}}_{vx} &= \hat{v}p_x - \hat{x}p_v, \\
\hat{\mathbf{J}}_{uv} &= \hat{u}p_v - \hat{v}p_u, \\
\hat{\mathbf{J}}_{ux} &= \hat{u}p_x - \hat{x}p_u, \\
\hat{\mathbf{J}}_{uy} &= \hat{u}p_y - \hat{y}p_{u,},
\end{aligned}\right\} \textit{where } p_x = -i\partial_x, \textit{etc.} \qquad (4.1)$$

These position and momentum operators then give us the Lie Algebra for E4. This Lie Algebra also applies to a larger covering group including non-zero spin representations (which is not analysed herein).

Lie Algebra

E4 contains 55 canonical commutation relations between the four position and six angular momentum operators. Taking care over permutations of indices, these fall into one of six types:

$$\begin{aligned}
&\left[\hat{\mathrm{J}}_{xu},\hat{\mathrm{J}}_{yu}\right]=i\hat{\mathrm{J}}_{xy} && (12\ \textit{relations}),\\
&\left[\hat{\mathrm{J}}_{xy},\hat{\mathrm{J}}_{uv}\right]=0 && (3\ \textit{relations}),\\
&\left[\hat{\mathrm{J}}_{xy},\hat{\mathrm{J}}_{xy}\right]=0 && (6\ \textit{relations}),\\
&\left[\hat{u},\hat{\mathrm{J}}_{xy}\right]=0 && (12\ \textit{relations}),\\
&\left[\hat{u},\hat{\mathrm{J}}_{vu}\right]=i\hat{v} && (12\ \textit{relations}),\\
&\left[\hat{u},\hat{u}\right]=\left[\hat{u},\hat{v}\right]=0 && (10\ \textit{relations}).
\end{aligned} \tag{4.2}$$

The angular momentum operators form an SO(4) sub algebra, isomorphic to SU(2) ⊕SU(2), and the position operators form an Abelian sub algebra. There are also four subsets of the $\hat{\mathrm{J}}_{xy}$ operators that form SO(3)/SU(2) sub algebras, comprising:

$$\begin{aligned}
&\left\{\hat{\mathrm{J}}_{uv},\hat{\mathrm{J}}_{vx},\hat{\mathrm{J}}_{ux}\right\},\\
&\left\{\hat{\mathrm{J}}_{uv},\hat{\mathrm{J}}_{vy},\hat{\mathrm{J}}_{uy}\right\},\\
&\left\{\hat{\mathrm{J}}_{xy},\hat{\mathrm{J}}_{yu},\hat{\mathrm{J}}_{xu}\right\},\\
&\left\{\hat{\mathrm{J}}_{xy},\hat{\mathrm{J}}_{yv},\hat{\mathrm{J}}_{xv}\right\}.
\end{aligned} \tag{4.3}$$

These sub-algebras are not independent, since any pair has non-commuting elements in common.

The two Casimir operators of E4 include $\hat{C}=\hat{u}^2+\hat{v}^2+\hat{x}^2+\hat{y}^2=1$, which represents the radius of the S^3 sphere, and also an analogue of the Pauli-Lubanski operator on the Lorentz-Poincare group (Jones, 1998), given by $\hat{W}^2=\hat{W}_x^2+\hat{W}_y^2+\hat{W}_u^2+\hat{W}_v^2$ where $W_x=\hat{y}\hat{J}_{vu}+\hat{v}\hat{J}_{uy}+\hat{u}\hat{J}_{yv}$, etc. We can also define a total angular momentum operator $\hat{\mathrm{J}}^2\equiv\hat{\mathrm{J}}_{xy}^2+\hat{\mathrm{J}}_{yv}^2+\hat{\mathrm{J}}_{vx}^2+\hat{\mathrm{J}}_{uv}^2+\hat{\mathrm{J}}_{ux}^2+\hat{\mathrm{J}}_{uy}^2$, which commutes with all the angular momentum operators, but not with the position operators:

$$\begin{aligned}
&\left[\hat{\mathrm{J}}^2,\hat{\mathrm{J}}_{uv}\right]=0 && (6\ \textit{relations}),\\
&\left[\hat{\mathrm{J}}^2,\hat{u}\right]=i\left\{\hat{\mathrm{J}}_{uv},\hat{v}\right\}+i\left\{\hat{\mathrm{J}}_{ux},\hat{x}\right\}+i\left\{\hat{\mathrm{J}}_{uy},\hat{y}\right\} && (4\ \textit{relations}).
\end{aligned} \tag{4.4}$$

We can thus choose a maximum of three commuting operators $\left\{\hat{\mathrm{J}}^2,\hat{\mathrm{J}}_{xy},\hat{\mathrm{J}}_{uv}\right\}$ to characterise the eigenstates of the system and label their respective quantum numbers by $|l,m,n\rangle$. The remaining momentum and position operators can be arranged into raising and lowering operator pairs. We introduce the four conjugate pairs of raising and lowering operators:

$$\begin{aligned}
&\hat{J}_{\alpha\pm}=\hat{\mathrm{J}}_{uy}-\hat{\mathrm{J}}_{xv}\pm i\left(\hat{\mathrm{J}}_{vy}+\hat{\mathrm{J}}_{xu}\right),\\
&\hat{J}_{\beta\pm}=\hat{\mathrm{J}}_{uy}+\hat{\mathrm{J}}_{xv}\pm i\left(\hat{\mathrm{J}}_{vy}-\hat{\mathrm{J}}_{xu}\right),\\
&U_{\pm}=\hat{u}\pm i\hat{v},\\
&X_{\pm}=\hat{x}\pm i\hat{y}
\end{aligned} \tag{4.5}$$

and obtain 55 restated commutation relations equivalent to (4.2):

$$\begin{aligned}
&\left[\hat{J}_{uv},\hat{U}_{\pm}\right]=\pm\hat{U}_{\pm}, \quad \left[\hat{\mathrm{J}}_{uv},\hat{J}_{\alpha\pm}\right]=\pm\hat{J}_{\alpha\pm}, \quad \left[\hat{\mathrm{J}}_{uv},\hat{J}_{\beta\pm}\right]=\pm\hat{J}_{\beta\pm},\\
&\left[\hat{J}_{xy},\hat{X}_{\pm}\right]=\pm\hat{X}_{\pm}, \quad \left[\hat{\mathrm{J}}_{xy},\hat{J}_{\alpha\pm}\right]=\pm\hat{J}_{\alpha\pm}, \quad \left[\hat{\mathrm{J}}_{xy},\hat{J}_{\beta\pm}\right]=-\pm\hat{J}_{\beta\pm},\\
&\left[\hat{J}_{\alpha+},\hat{J}_{\alpha-}\right]=4\left(\hat{\mathrm{J}}_{uv}+\hat{\mathrm{J}}_{xy}\right), \quad \left[\hat{J}_{\alpha+},\hat{U}_{-}\right]=2\hat{X}_{+}, \quad \left[\hat{J}_{\alpha+},\hat{X}_{-}\right]=-2\hat{U}_{+},\\
&\left[\hat{J}_{\beta+},\hat{J}_{\beta-}\right]=4\left(\hat{\mathrm{J}}_{uv}-\hat{\mathrm{J}}_{xy}\right), \quad \left[\hat{J}_{\beta+},\hat{U}_{-}\right]=-2\hat{X}_{-}, \quad \left[\hat{J}_{\beta+},\hat{X}_{+}\right]=2\hat{U}_{+},\\
&\left[\hat{J}_{\alpha-},\hat{U}_{+}\right]=-2\hat{X}_{-}, \quad \left[\hat{J}_{\alpha-},\hat{X}_{+}\right]=2\hat{U}_{-},\\
&\left[\hat{J}_{\beta-},\hat{U}_{+}\right]=2\hat{X}_{+}, \quad \left[\hat{J}_{\beta-},\hat{X}_{-}\right]=-2\hat{U}_{-},\\
&\left[\hat{J}_{uv},\hat{\mathrm{J}}_{uv}\right]=\left[\hat{\mathrm{J}}_{xy},\hat{\mathrm{J}}_{xy}\right]=\left[\hat{\mathrm{J}}_{uv},\hat{\mathrm{J}}_{xy}\right]=\left[\hat{J}_{\alpha-},\hat{J}_{\alpha-}\right]=\left[\hat{J}_{\alpha+},\hat{J}_{\alpha+}\right]=\left[\hat{J}_{\beta-},\hat{J}_{\beta-}\right]=\\
&\left[\hat{J}_{\beta+},\hat{J}_{\beta+}\right]=\left[\hat{J}_{uv},\hat{X}_{\pm}\right]=\left[\hat{J}_{xy},\hat{U}_{\pm}\right]=\left[\hat{J}_{\alpha\pm},\hat{J}_{\beta\pm}\right]=\left[\hat{J}_{\alpha+},\hat{U}_{+}\right]=\left[\hat{J}_{\alpha+},\hat{X}_{+}\right]=\\
&\left[\hat{J}_{\beta+},\hat{U}_{+}\right]=\left[\hat{J}_{\beta+},\hat{X}_{-}\right]=\left[\hat{J}_{\alpha-},\hat{U}_{-}\right]=\left[\hat{J}_{\alpha-},\hat{X}_{-}\right]=\left[\hat{J}_{\beta-},\hat{U}_{-}\right]=\left[\hat{J}_{\beta-},\hat{X}_{+}\right]=\\
&\left[\hat{U}_{\pm},\hat{U}_{\pm}\right]=\left[\hat{U}_{\pm},\hat{X}_{\pm}\right]=\left[\hat{X}_{\pm},\hat{X}_{\pm}\right]=0.
\end{aligned} \tag{4.6}$$

Since the U and X operators do not commute with J^2, we can summarise the commutation relations in the following raising and lowering operator actions (which

have yet to be normalised and where we have yet to determine the action on *l* of U and X):

$$\begin{array}{c|c} \hat{O} & \hat{O}|l,m,n\rangle \\ \hline \hat{J}_{\alpha+} & |l,m+1,n+1\rangle \\ \hat{J}_{\beta+} & |l,m-1,n+1\rangle \\ \hat{J}_{\alpha-} & |l,m-1,n-1\rangle \\ \hat{J}_{\beta-} & |l,m+1,n-1\rangle \\ \mathrm{U}_{\pm} & |l',m,n\pm 1\rangle \\ \mathrm{X}_{\pm} & |l',m\pm 1,n\rangle \end{array} . \tag{4.7}$$

These four pairs of raising and lowering operators hold the key to finding an eigenstate basis for E4 and to understanding the pattern of eigenvalue perturbations due to a potential. In particular, we note that the $\hat{J}_{\pm}$ raising and lowering operators either leave the sum m+n unchanged or change it by a double integer step.

Basis for Eigenstates

Similarly to the case of E3, the E4 algebra admits angular coordinate representations, one of which is given by the mapping:

$$\begin{aligned} \hat{u} &\to \cos\psi, \\ \hat{v} &\to \sin\psi\cos\theta, \\ \hat{x} &\to \sin\psi\sin\theta\cos\phi, \\ \hat{y} &\to \sin\psi\sin\theta\sin\phi. \end{aligned} \tag{4.8}$$

We shall study systems characterised by zero spin (W^2=0), in which case we can obtain the angular momentum operators by pulling back the orbital angular momentum operators from $\mathbb{R}^4$ to S^3. This leads to the operator representations:

$$
\begin{aligned}
&U_{\pm} = \cos\psi \pm i\sin\psi\cos\theta,\\
&X_{\pm} = \sin\psi\sin\theta e^{\pm i\phi},\\
&\hat{\mathrm{J}}_{xy} = \hat{x}p_y - \hat{y}p_x = -i\left(\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\phi},\\
&\hat{\mathrm{J}}_{uv} = \hat{u}p_v - \hat{v}p_u = -i\left(\hat{u}\frac{\partial}{\partial v} - \hat{v}\frac{\partial}{\partial u}\right) = -i\left(\cos\theta\frac{\partial}{\partial\psi} - \frac{\cos\psi\sin\theta}{\sin\psi}\frac{\partial}{\partial\theta}\right),\\
&\hat{J}_{\alpha+} = e^{i\phi}\left[\frac{\sin\psi\cos\theta - i\cos\psi}{\sin\psi\sin\theta}\frac{\partial}{\partial\phi} - \left(\frac{\cos\psi\cos\theta}{\sin\psi} + i\right)\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial}{\partial\psi}\right],\\
&\hat{J}_{\alpha-} = e^{-i\phi}\left[-\frac{\sin\psi\cos\theta + i\cos\psi}{\sin\psi\sin\theta}\frac{\partial}{\partial\phi} + \left(\frac{\cos\psi\cos\theta}{\sin\psi} - i\right)\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial\psi}\right],\\
&\hat{J}_{\beta+} = e^{-i\phi}\left[\frac{\sin\psi\cos\theta - i\cos\psi}{\sin\psi\sin\theta}\frac{\partial}{\partial\phi} + \left(\frac{\cos\psi\cos\theta}{\sin\psi} + i\right)\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial\psi}\right],\\
&\hat{J}_{\beta-} = e^{i\phi}\left[-\frac{\sin\psi\cos\theta + i\cos\psi}{\sin\psi\sin\theta}\frac{\partial}{\partial\phi} - \left(\frac{\cos\psi\cos\theta}{\sin\psi} - i\right)\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial}{\partial\psi}\right].
\end{aligned}
\tag{4.9}
$$

These naturally satisfy the commutation relations (4.6) and allow us to form the total angular momentum operator or Laplacian:

$$
\hat{J}^2 = -\frac{1}{\sin^2\psi}\left(\frac{\partial}{\partial\psi}\sin^2\psi\frac{\partial}{\partial\psi} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \equiv -\nabla^2_{\psi\theta\phi}. \tag{4.10}
$$

This enables us to construct a Hamiltonian for E4 and we can proceed first to find its free eigenstates and their eigenvalues spectrum and then to explore its perturbation under a potential. An eigenstate of E4 characterised by the $|l,m,n\rangle$ quantum labels introduced above must satisfy the following relations:

$$
\begin{aligned}
&\hat{\mathrm{J}}_{xy}\Psi = -i\frac{\partial}{\partial\phi}\Psi = m\Psi,\\
&\hat{\mathrm{J}}_{uv}\Psi = -i\left(\cos\theta\frac{\partial}{\partial\psi} - \frac{\cos\psi\sin\theta}{\sin\psi}\frac{\partial}{\partial\theta}\right)\Psi = n\Psi.
\end{aligned}
\tag{4.11}
$$

With a little help from *Mathematica*, we find this imposes solutions of the form:

$$\Psi = F\left(\sin\theta\sin\psi\right)\left(\frac{\cos\psi + i\,\cos\theta\,\sin\psi}{\cos\psi - i\cos\theta\,\sin\psi}\right)^{n/2} e^{im\phi} \tag{4.12}$$

where F is an arbitrary function of $\sin\theta\sin\phi$ and m must be an integer if we require 2π periodicity over ϕ. We can impose further constraints on the solutions by requiring that the eigenstates should form finite representations. Suppose we define our top state Ψ_{top} as one that is annihilated by both $\hat{J}_{\alpha+}$ and $\hat{J}_{\beta+}$, so that $\hat{J}_{\alpha+}\Psi_{top} = \hat{J}_{\beta+}\Psi_{top} = 0$. By combining (4.9) and (4.12), we obtain:

$$\begin{aligned}
&\hat{J}_{\alpha+} : -m\left[1-\left(\sin\psi\sin\theta\right)^2\right] + n\left(\sin\psi\sin\theta\right)^2 + \sin\psi\sin\theta\left[1-\left(\sin\psi\sin\theta\right)^2\right]\frac{F'}{F} = 0,\\
&\hat{J}_{\beta+} : m\left[1-\left(\sin\psi\sin\theta\right)^2\right] + n\left(\sin\psi\sin\theta\right)^2 + \sin\theta\sin\psi\left[1-\left(\sin\psi\sin\theta\right)^2\right]\frac{F'}{F} = 0.
\end{aligned} \tag{4.13}$$

These can only both be non-trivially satisfied in the case m=0, whence:

$$\begin{aligned}
&\frac{dF}{F} = \frac{-n\left(\sin\psi\sin\theta\right)d\left(\sin\psi\sin\theta\right)}{\left(1-\sin\psi^2\sin^2\theta\right)},\\
&F \propto \left(1-\sin\psi^2\sin^2\theta\right)^{\frac{n}{2}},\\
&\Psi_{top} \propto \left(\cos\psi + i\,\cos\theta\,\sin\psi\right)^n.
\end{aligned} \tag{4.14}$$

Similarly, by requiring $\hat{J}_{\alpha-}\Psi_{bottom} = \hat{J}_{\beta-}\Psi_{bottom} = 0$, we obtain $\Psi_{bottom} \propto \left(\cos\psi + i\,\cos\theta\,\sin\psi\right)^{-n}$. We take n to be an integer, so that irreps close under the raising and lowering operators in this zero spin representation. Thus, each top state value of n, which we can define as *l*, specifies a multiplet containing several m and n states. In each multiplet, m and n both vary between $+l$ and $-l$ and their permissible values are restricted further by the available raising/lowering operations. **Table 2** sets out the first four E4 multiplets of S^3 spherical harmonics for *l* values varying from 0 to

3. The eigenvalues of the total angular momentum operator J^2 follow an $l(l+2)$ series, being degenerate with respect to m and n.

Table 2: E4 S^3 spherical harmonics for l=0 to 3 (W^2=0 representation)

J^2	l	m	n	$\Psi_{l,m,n}(\psi,\theta,\phi)$
$l(l+2)$		$\lvert m\rvert < l$	$\lvert n\rvert < l$	$\left(\cos\psi + (\lvert n\rvert / n)\, i\cos\theta\sin\psi\right)^{\lvert n\rvert} \sin^{\lvert m\rvert}\theta\ \sin^{\lvert m\rvert}\psi e^{im\phi}$
0	0	0	0	1
3	1	±1	0	$\sin\theta\ \sin\psi\exp(\pm i\phi)$
3	1	0	±1	$\cos\psi \pm i\cos\theta\sin\psi$
8	2	±2	0	$\sin^2\theta\ \sin^2\psi\exp(\pm 2i\phi)$
8	2	±1	±1	$(\cos\psi \pm_n i\cos\theta\sin\psi)\sin\theta\ \sin\psi\exp(\pm_m i\phi)$
8	2	0	±2	$(\cos\psi \pm i\cos\theta\sin\psi)^2$
8	2	0	0	$1-2\sin^2\theta\sin^2\psi$
15	3	±3	0	$\sin\theta\ \sin\psi\exp(\pm i3\phi)$
15	3	±2	±1	$(\cos\psi \pm_n i\cos\theta\sin\psi)\sin^2\theta\ \sin^2\psi\exp(\pm_m 2i\phi)$
15	3	±1	±2	$(\cos\psi \pm_n i\cos\theta\sin\psi)^2\sin\theta\ \sin\psi\exp(\pm_m i\phi)$
15	3	0	±3	$(\cos\psi \pm i\cos\theta\sin\psi)^3$
15	3	±1	0	$(2-3\sin^2\theta\sin^2\psi)\sin\theta\ \sin\psi\exp(\pm i\phi)$
15	3	0	±1	$(1-3\sin^2\theta\sin^2\psi)(\cos\psi \pm i\cos\theta\sin\psi)$

It is helpful to normalise the eigenstates $|l,m,n\rangle$ relative to the top state of each multiplet, taken as $|l,0,l\rangle$. The normalisation follows from the commutation relations (4.6) and is calculated in the Appendix Normalisation of E4 Eigenstates, adapting established methodology for the Lorentz Group (Naimark, 1964). We obtain the result:

$$|l,m,n\rangle = N_{lmn}\ \hat{J}_{\alpha-}^{(l-m-n)/2}\ \hat{J}_{\beta-}^{(l+m-n)/2}\,|l,0,l\rangle, \tag{4.15}$$

$$N_{lmn} = \frac{1}{2^{l-n}\,l!}\sqrt{\frac{\left(\frac{l+m+n}{2}\right)!\left(\frac{l-m+n}{2}\right)!}{\left(\frac{l-m-n}{2}\right)!\left(\frac{l+m-n}{2}\right)!}}. \tag{4.16}$$

Given the numerous spatial dimensions available, there exist alternative choices for the Parity transformation including:

$$\begin{aligned}\mathcal{P}(\hat{v}):&\ (\hat{u},\hat{v},\hat{x},\hat{y})\to(\hat{u},-\hat{v},\hat{x},\hat{y}),\\ &\ \psi\to-\psi;\\ \mathcal{P}(\hat{y}):&\ (\hat{u},\hat{v},\hat{x},\hat{y})\to(\hat{u},\hat{v},\hat{x},-\hat{y}),\\ &\ \phi\to-\phi.\end{aligned} \tag{4.17}$$

By choosing the appropriate reflection, we can see that a number, but not all, of the states are PT invariant. Each multiplet is however PT invariant, with Parity and Time Reversal transformations reversing the m and/or n values of the states.

Hermitian and PT Symmetric Hamiltonians

As for E2 and E3, we can construct PT symmetric Hamiltonians in E4 from imaginary potentials oriented along one of the coordinate axes and also from the raising and lowering operators. The directional potentials can also be expressed as linear combinations of the raising and lowering operators: $\hat{u}=\frac{1}{2}\left(\hat{U}_{+}+\hat{U}_{-}\right)$, etc. Thus, we need to find the matrix elements for these raising and lowering operators relative to the eigenstate basis and then diagonalise the appropriate Hamiltonian matrix. We define a general perturbed state using the E4 basis states as $\Psi=\sum_{l',m',n'}a_{l'm'n'}\left|l',m',n'\right\rangle$ and obtain the eigenvalues E of perturbed states from the solutions to:

$$\underbrace{\sum_{l',m',n'}\underbrace{l(l+2)\delta_{l,l'}\delta_{m,m'}\delta_{n,n'}+g\underbrace{\langle l,m,n|\hat{u}|l',m',n'\rangle}_{matrix\ element}}_{Hamiltonian\ matrix}a_{l'm'n'}}=Ea_{l,m,n}. \tag{4.18}$$

We first evaluate the individual non-zero matrix elements analytically, following calculations set out in the Appendix Calculation of E4 Matrix Elements, which adapts established methodology (Naimark, 1964), to find:

$$\begin{aligned}
\langle l+1,m,n+1|\hat{u}|l,m,n\rangle &= \frac{1}{4}\sqrt{\frac{(l+m+n+2)(l-m+n+2)}{(l+1)(l+2)}},\\
\langle l-1,m,n+1|\hat{u}|l,m,n\rangle &= \frac{1}{4}\sqrt{\frac{(l+m-n)(l-m-n)}{l(l+1)}},\\
\langle l-1,m,n-1|\hat{u}|l,m,n\rangle &= \frac{1}{4}\sqrt{\frac{(l+m+n)(l-m+n)}{l(l+1)}},\\
\langle l+1,m,n-1|\hat{u}|l,m,n\rangle &= \frac{1}{4}\sqrt{\frac{(l+m-n+2)(l-m-n+2)}{(l+1)(l+2)}}.
\end{aligned} \tag{4.19}$$

We arrange these elements into a Hamiltonian matrix by taking the multiplets in turn, starting from l=0 (1 component), l=1 (4 components), to some chosen maximum *l*-value, scaling by g and adding the diagonal terms. Since the $g\hat{u}$ or g cosψ potential does not mix m-states, we only need to enumerate over the n states in each l-multiplet for any fixed value of m. The matrix (4.20) shows the l=0,1,2 multiplets from a Hamiltonian matrix, where we have chosen m=0 and assigned signs using the Condon and Shortley phase convention for m and n states.

$\langle l',m',n'\|\hat{H}\|l,m,n\rangle$	$\|0,0,0\rangle$	$\|1,0,-1\rangle$	$\|1,0,1\rangle$	$\|2,0,-2\rangle$	$\|2,0,0\rangle$	$\|2,0,2\rangle$
$\langle 0,0,0\|$	0	$-\frac{g}{2\sqrt{2}}$	$\frac{g}{2\sqrt{2}}$	0	0	0
$\langle 1,0,-1\|$	$-\frac{g}{2\sqrt{2}}$	3	0	$-\frac{g}{\sqrt{6}}$	$-\frac{g}{2\sqrt{6}}$	0
$\langle 1,0,1\|$	$\frac{g}{2\sqrt{2}}$	0	3	0	$\frac{g}{2\sqrt{6}}$	$\frac{g}{\sqrt{6}}$
$\langle 2,0,-2\|$	0	$-\frac{g}{\sqrt{6}}$	0	8	0	0
$\langle 2,0,0\|$	0	$-\frac{g}{2\sqrt{6}}$	$\frac{g}{2\sqrt{6}}$	0	8	0
$\langle 2,0,2\|$	0	0	$\frac{g}{\sqrt{6}}$	0	0	8

(4.20)

The matrix is symmetric and consists of tridiagonal blocks linking adjacent l-multiplets. The Hamiltonian is manifestly Hermitian if g is real and is PT symmetric (under {n,i}→

{-n,-i}) if g is imaginary. For large *l*, the off-diagonal terms tend to g/4, while the diagonal terms grow as l^2. Consequently, for large *l* the matrix effectively becomes diagonal with minimal mixing, consistent with our approach of diagonalisation based on the lowest *l*-multiplets. The numerical calculations in the next section draw on matrices containing up to 10 *l*-multiplets.

Eigenvalue Splitting Patterns: $g\hat{u}$ or $g\cos\psi$ potential

The shifting of E4 energy levels in the first four l-multiplets for a Hermitian potential with g real is shown in Figure 20. The eigenvalues in each l-multiplet are split by n, but remain degenerate with respect to m. The eigenvalues are shifted from the l(l+2) series at g=0, but remain real, and their associated eigenstates remain distinct. The mixing of the eigenvalues of the same four l-multiplets when subjected to an imaginary PT-symmetric potential is shown in Figure 21 and Figure 22.

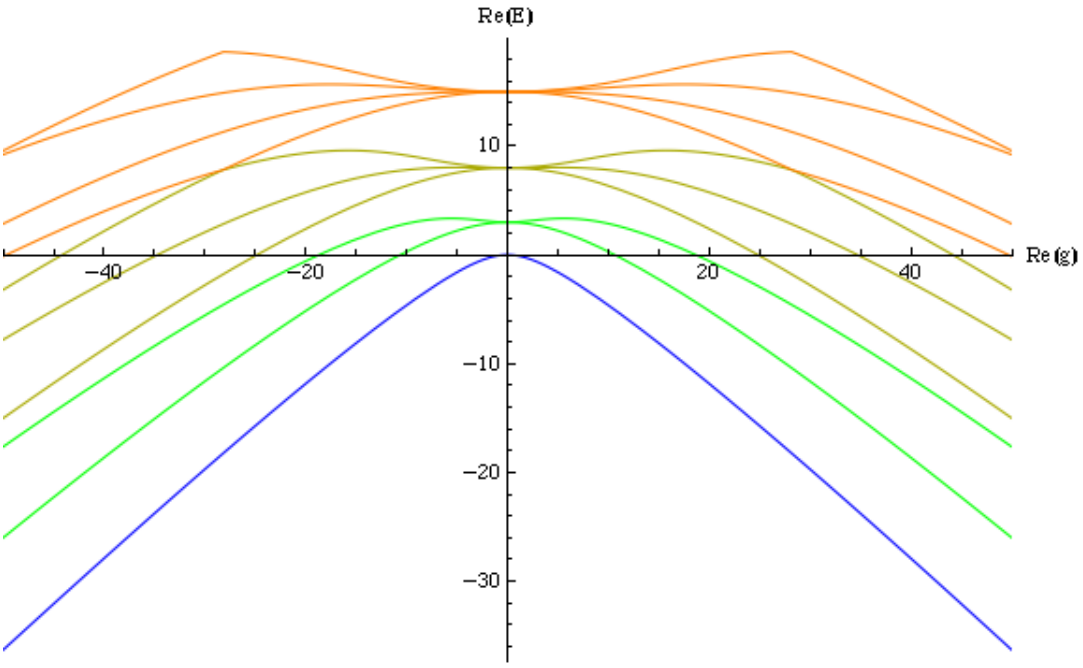

Figure 20: Eigenvalues for E4 bosonic (W^2=0) states under a real g cosψ potential for l=0 to 3. The l-multiplets are degenerate with respect to m but split according to n. The eigenvalues are real with intercepts on the E axis given by the *l(l+2)* series.

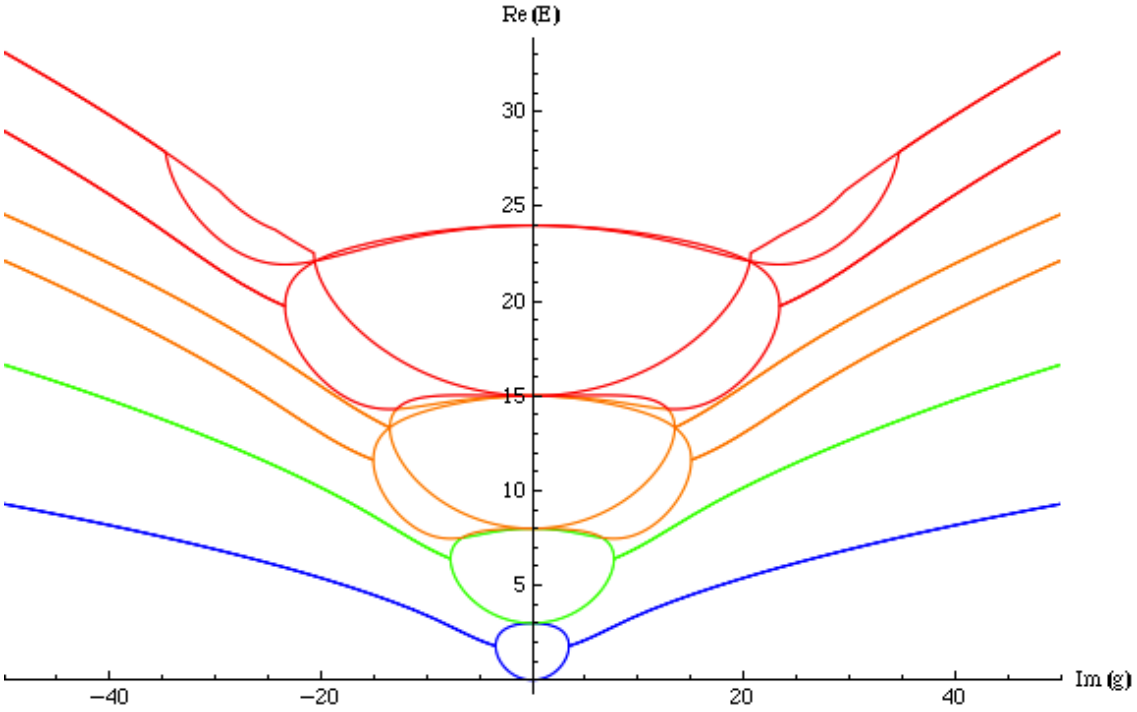

Figure 21: Real components of eigenvalues for E4 bosonic (W^2=0) states under an imaginary g cosψ potential for *l*=0 to 4. Intercepts on the E axis are given by the *l(l+2)* series.

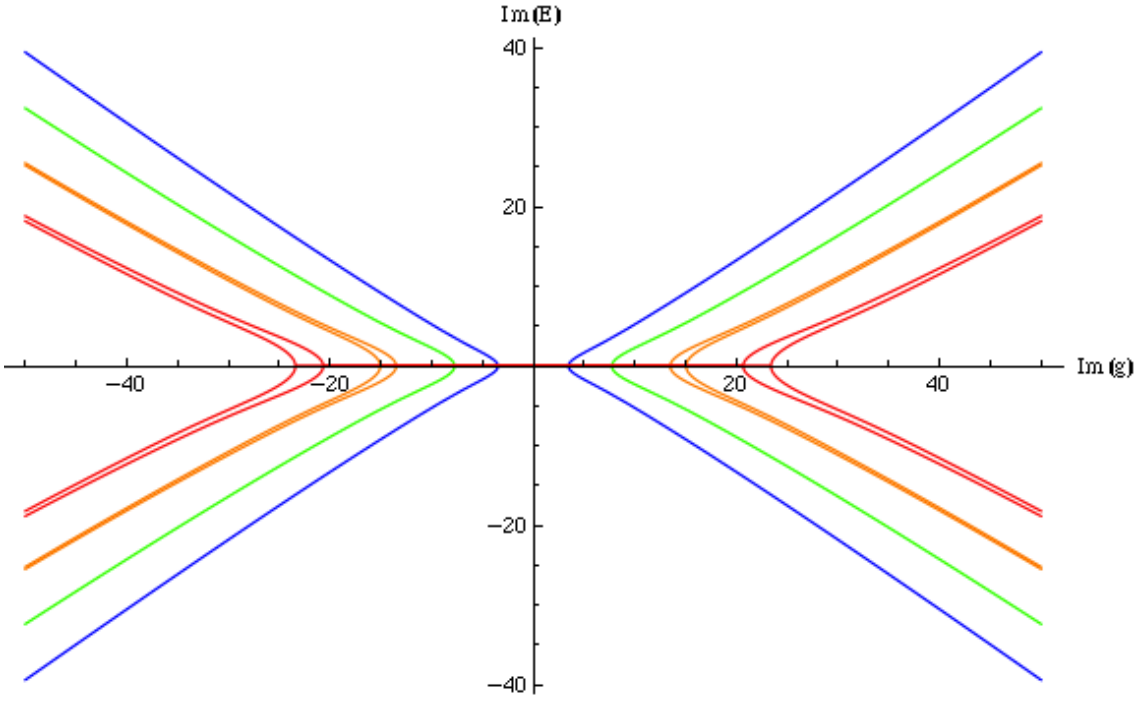

Figure 22: Imaginary components of eigenvalues for E4 bosonic (W^2=0) states under an imaginary g cosψ potential for *l*=0 to 4. There is a region of unbroken PT symmetry before eigenvalues from different *l*-multiplets combine into complex conjugate pairs.

The patterns for the E4 bosonic (W^2=0) series are thus similar to those for the E2 periodic and E3 spin zero series. Pairs of real eigenstates from adjacent multiplets are mixed by an imaginary directional potential, with their eigenvalues equalising at a critical value of the potential for each pair, as summarised in (4.21). As the potential is increased beyond these critical points, the eigenvalues form complex conjugate pairs.

Multiplet Pairs $\{l_1,l_2\}$	*Critical Value* g
$\{0,1\}$	3.4645 I
$\{1,2\}$	7.7374 I
$\{2,3\}$	13.4833 I
$\{2,3\}$	15.0490 I
$\{3,4\}$	20.6662 I

(4.21)

The mixing of the first pair occurs at a critical value of Im(g)= 3.4645. Above this value, PT symmetry is broken and only some of the eigenvalues remain real. Below this value all the eigenvalues are real and we have a region of unbroken PT symmetry.

Other potentials

Due to the S^3 symmetries of E4, potentials based on any of the $\hat{u},\hat{v},\hat{x},\hat{y}$ behave in a similar manner to each other (in the absence of spin) and when coupled to the free Hamiltonian by an imaginary parameter g, give rise to PT symmetric potentials that have regions of unbroken PT symmetry. For example, a switch between $\hat{u}$ and $\hat{x}$ corresponds simply to the interchange of m and n quantum numbers.

As for E2 and E3, the $\hat{X}_{\pm}$ raising and lowering operators can be obtained from the $\hat{x}$ operator from the coordinate similarity transformation $\phi \to \phi + i\alpha$. The $\hat{X}_{\pm}$, $\hat{U}_{\pm}$, $\hat{J}_{\alpha\pm}$ and $\hat{J}_{\beta\pm}$ raising and lowering operators are all PT symmetric and have unbounded parametric regions of unbroken PT symmetry.

Finally, potentials of the form $V = g\hat{J}_{xy}$ and $V = g\hat{J}_{uv}$ lead to diagonal Hamiltonians, which are Hermitian if g is real but PT symmetric if g is imaginary. These Hamiltonians have diagonal matrices, with the result that if g is imaginary, then the eigenvalues are generally complex, and exhibit regions of unbroken PT symmetry only for m=0 and n=0 states respectively, for which the potentials vanish.

Key Findings: E4

We have shown that we can construct non-Hermitian, but PT symmetric Hamiltonians on the E4 manifold, where $\hat{J}^2$ is the Laplacian operator on the E4 manifold defined in (4.10):

$$\begin{aligned}
&\mathcal{H}(\hat{u}) = \hat{J}^2 + g\ \hat{u} && (or\ \hat{v}\ or\ \hat{x}\ or\ \hat{y}) && (g\ imaginary),\\
&\mathcal{H}(U_{\pm}) = \hat{J}^2 + g\ \hat{U}_{\pm} && \left(or\ \hat{X}_{\pm}\ \right) && (g\ real\ or\ imaginary),\\
&\mathcal{H}\left(\hat{J}_{\alpha\pm}\right) = \hat{J}^2 + g\ \hat{J}_{\alpha\pm} && \left(or\ \hat{J}_{\beta\pm}\ \right) && (g\ real\ or\ imaginary),\\
&\mathcal{H}\left(\hat{J}_{xy}\ \right) = \hat{J}^2 + g\ \hat{J}_{xy} && \left(or\ \hat{J}_{uv}\ \right) && (g\ imaginary).
\end{aligned} \tag{4.22}$$

PT symmetric Hamiltonians $\mathcal{H}(\hat{u})$ exhibit regions of unbroken PT symmetry for the bosonic spin zero (W^2=0) series for imaginary values of g below critical points. We have not studied their behaviour in non-zero spin representations. PT symmetric Hamiltonians $\mathcal{H}\left(\hat{U}_{\pm}\right)$ and $\mathcal{H}\left(\hat{J}_{\pm}\right)$, which are based on raising or lowering operators, are upper or lower triangular and have unbounded regions of unbroken PT symmetry. PT symmetric Hamiltonians $\mathcal{H}\left(\hat{J}_{xy}\ \right)$ or $\mathcal{H}\left(\hat{J}_{uv}\ \right)$ have regions of unbroken PT symmetry only for m=0 (or n=0) quantum states. As before, these Hamiltonians can naturally be generalised using rotational symmetries (subject to spin) and should be viewed as representatives of equivalence classes.

Chapter Five

Discussion and Conclusions

Euclidean Group PT symmetric potentials

This study of the PT symmetry of E2, E3 and E4 has covered a rich landscape of Lie Algebras and their representations in terms of quantum mechanical operators and basis states. Working with position and momentum operators, we have identified simple Hermitian and non-Hermitian but PT symmetric Hamiltonians that can easily be constructed using building blocks from the Lie Algebras. The table below summarises the key findings of the detailed analysis of E2, E3 and E4.

Table 3: PT Symmetry of Lie Algebra operators on Euclidean Manifolds

Operators	*Type*	*E2*	*E3*	*E4*
Position	*Hermitian*	$g\hat{u}, g\hat{v}$	$g\hat{u}, g\hat{v}, g\hat{w}$	$g\hat{u}, g\hat{v}, g\hat{x}, g\hat{y}$
	PT Symmetric	$ig\hat{u}, ig\hat{v}$	$ig\hat{u}, ig\hat{v}, ig\hat{w}$	$ig\hat{u}, ig\hat{v}, ig\hat{x}, ig\hat{y}$
	Unbroken PTS	*Periodic* : $g < g_{crit}$	$s = 0^*$: $g < g_{crit}$	$W^2 = 0$: $g < g_{crit}$
Momentum	*Hermitian*	$g\hat{J}$	$g\hat{J}_w$	$g\hat{J}_{xy}, g\hat{J}_{uv}$
	PT Symmetric	$ig\hat{J}$	$ig\hat{J}_w$	$ig\hat{J}_{xy}, ig\hat{J}_{uv}$
	Unbroken PTS	$m = 0$ *(trivial)*	$m = 0$	$m = 0, n = 0$
Raising / Lowering	*Hermitian*	-	-	-
	PT Symmetric	$\hat{W}_\pm$	$\hat{W}_\pm, \hat{J}_\pm$	$\hat{X}_\pm, \hat{U}_\pm, \hat{J}_{\alpha\pm}, \hat{J}_{\beta\pm}$
	Unbroken PTS	*All*	*All*	*All*

g is taken as real throughout

** : there are also regions of unbroken PT Symmetry for $m = 0$, s integer states*

There are clear relationships between the properties of the higher and lower dimensioned algebras. This is unsurprising since S^1 is a sub-manifold of S^2, which is in turn a sub-manifold of S^3 and the rotation groups are related in a hierarchy SO(2)⊂ SO(3) ⊂ SO(4) ≅ SO(3) ⊕ SO(3). We can reasonably conjecture therefore that the

general properties identified herein will carry over to Euclidean Group manifolds of higher dimension.

We can shed some theoretical light on the conditions under which PT symmetric Hamiltonians exhibit regions of unbroken PT symmetry by analysis of the structure (referred to the free eigenstate basis) of the Hamiltonian matrices that incorporate the perturbing potential.

Firstly, consider the case of PT symmetric potentials containing the momentum eigenvalue operators. These are already diagonal and therefore the reality of the eigenvalues is determined by the reality of the coupling parameter g. Regions of unbroken PT symmetry can only arise for states for which the perturbing operator has zero eigenvalues and only these states thus have energy eigenvalues which are unchanged from the free basis.

Secondly, consider the case of PT symmetric potentials incorporating a single raising or lowering operator. These operators shift quantum numbers in one direction only and therefore, by orthogonality of the basis states, have Hamiltonian matrices that are either upper or lower triangular, with the free basis eigenvalues on the diagonal. Thus, the system eigenvalues are unchanged from the free basis under diagonalisation, even though mixing of basis eigenstates occurs.

The complicated case is that of PT symmetric potentials incorporating a position operator multiplied by an imaginary parameter g to give a directional potential. Regardless of the spin representation, the PT symmetric Hamiltonians in this latter category are represented by symmetric matrices with imaginary off-diagonal terms and have a block tridiagonal structure, with each block containing the degenerate basis states for each energy or total angular momentum quantum number *l*.

One important principle is that in order for a Hamiltonian to have entirely real eigenvalues, its matrix must have a real trace (since the trace is preserved by the similarity transformation required to diagonalise the matrix). This means that, in order for the matrix to have entirely real eigenvalues, the imaginary g parameter should not appear in the trace. The requirement that the trace of the mixing matrix should be real imposes a necessary condition for unbroken PT symmetry that any imaginary potential should trace to zero. Put in other words, if a directional potential maps basis states to themselves, then it will not generate unbroken PT symmetry. This can be seen, for example, in **Figure 14** and **Figure 15**, where the s=1, m=1 state is mixed with itself by a directional potential, producing complex eigenvalues for an arbitrarily small imaginary coupling parameter.

Reality of the trace is a necessary, but not a sufficient condition. In particular, we have the observation that fermionic representations do not exhibit regions of unbroken PT symmetry under imaginary directional potentials, even when their Hamiltonian traces are real.

PT Symmetry of Bosons vs Fermions

Why is it then that some PT symmetric Hamiltonian matrices with imaginary off-diagonal terms have real eigenvalues while others do not?

We can shed some light by considering the different structure of the group representations between the half integer fermionic and integer bosonic series. For example, in the case of the integer series, the irreps all have an odd number of states, while the half-integer irreps have an even number of states. Thus while the basis states in half-integer series come in degenerate pairs, the integer series contain unpaired basis states. Consider the difference between the following two matrix diagonalisation

problems, which have been simplified from those for E2 Hamiltonians with PT symmetric directional potentials:

$$Diagonalise: \begin{pmatrix} 1 & ig \\ ig & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1+ig & 0 \\ 0 & 1-ig \end{pmatrix}, \tag{5.1}$$

$$Diagonalise: \begin{pmatrix} 1 & ig & 0 \\ ig & 0 & ig \\ 0 & ig & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\left(1+\sqrt{1-8g^2}\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(1-\sqrt{1-8g^2}\right) \end{pmatrix}. \tag{5.2}$$

In the case of (5.1), which is a truncation of a half-integer E2 Hamiltonian, diagonalisation always gives a complex conjugate pair of eigenvalues for an arbitrarily small parameter g, whereas in the case of (5.2), which is a truncation of an integer E2 Hamiltonian, the diagonalisation only yields complex conjugate pairs for values of g greater than $1/\sqrt{8}$. We observe, for example in Figure 3, the lowest two non-degenerate integer eigenvalues forming a degenerate pair before splitting into a complex conjugate pair. On the other hand, as can be seen in Figure 6, the half-integer states start as degenerate pairs and therefore split immediately.

Differences in the PT symmetric behaviour of periodic and antiperiodic functions have also been observed in the study of harmonic potentials on the manifold $\mathbb{R}$ (Bender et al., 1998). Presumably the mechanism at work is similar, since the quantum mechanics on S^1 reduces to that on $\mathbb{R}$ as the radius of the circle is increased to infinity (Isham, 1983).

Recent work on the PT symmetric quantum field theory of fermionic functions (Bender and Klevansky, 2011), (Jones-Smith and Mathur, 2010) considers whether the time reversal operator $\mathcal{T}$ should be endowed with the property $\mathcal{T}^2 = -I$ for fermions. It has

been shown (Brink and Satchler, 1968) that *if one requires a wave function to be invariant* under an anti-linear time reversal operator θ, then $\theta^2 = I$ if the wave function has integral angular momentum and $\theta^2 = -I$ if the wavefunction has half-integral angular momentum, and that an operator which produces such an effect is given by $\theta|l,m\rangle \equiv (-1)^{l-m}|l,-m\rangle$. However, we do not require the invariance of wavefunctions under time reversal in this study and therefore we may just use the time reversal operator $\mathcal{T}$ as defined by complex conjugation as in (1.5).

Relationship of PT Symmetric Hamiltonians to Klein Gordon Equation

Before moving to draw conclusions in relation to PT symmetry from this study of Euclidean manifolds, it is appropriate to digress on the subject of time, which has not been treated in coordinate terms, manifesting itself merely through complex conjugation. We can, alternatively, incorporate time in a more explicit manner by relating the Hamiltonian equations that we have been studying to Klein Gordon equations on underlying curved spacetime manifolds.

The Klein Gordon equation for a field Ψ, where $\Box$ represents the d'Alembertian operator is:

$$\Box\Psi - m^2\Psi = 0\,. \tag{5.3}$$

Consider a static manifold that permits a spacelike foliation $\Sigma \times \mathbb{R}$, with Σ representing its spatial element and $\mathbb{R}$ representing time, that has a metric of the type:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \gamma_{ij} \end{pmatrix}, \tag{5.4}$$

where γ_{ij} represents the spatial element of the metric and γ_{ij} has determinant γ. We can expand the Klein Gordon equation on this manifold and decouple the spatial Laplacian from the time coordinate:

$$\nabla^2\Psi - \partial_0^2\ \Psi = m^2\Psi\,. \tag{5.5}$$

If we look for harmonic solutions of the type $\Psi(\mathbf{x},t) = \psi(\mathbf{x})\ e^{-i\omega t}$, then (5.5) assumes the form:

$$\underbrace{-\nabla^2}_{\mathcal{H}}\psi(\mathbf{x}) = -\frac{1}{\sqrt{\gamma}}\partial_i\gamma^{ij}\sqrt{\gamma}\partial_j\psi(\mathbf{x})\ = \underbrace{\left(\omega^2 - m^2\right)}_{E}\psi(\mathbf{x})\,. \tag{5.6}$$

We can thus recognise the total angular momentum Hamiltonian operator $\mathcal{H}$ and the energy eigenvalues E that we have been dealing with in the cases of the unperturbed E2, E3 and E4 systems. Table 4 makes the correspondence between the Hamiltonians we have studied and the metrics that generate them (through the Klein Gordon equation) more explicit.

Table 4: Metrics on Euclidean Group Manifolds

	$metric\ ds^2$	$\mathcal{H}=\hat{\mathbf{J}}^2$	E
$E2$	$-dt^2 + d\theta^2$	$-\dfrac{\partial^2}{\partial\theta^2}$	m^2
$E3$ $(s=0)$	$-dt^2 + d\theta^2 + \sin^2\theta d\phi^2$	$-\nabla^2_{\theta\phi}$	$l(l+1)$
$E3$ $(s\neq 0)$	$-dt^2$ $+d\theta^2 + d\phi^2 + d\chi^2 - 2\cos\theta d\phi d\chi$	$-\nabla^2_{\theta\phi}\cdots$ $+2\dfrac{\cos\theta}{\sin^2\theta}\dfrac{\partial}{\partial\phi}\dfrac{\partial}{\partial\chi} - \dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\chi^2}$	$l(l+1)$
$E4$	$-dt^2$ $+d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi\sin^2\theta d\phi^2$	$-\nabla^2_{\theta\phi\psi}$	$l(l+2)$

In the absence of spin, the Laplacian follows naturally from the spatial metric through the total orbital angular momentum operator representation. The free Hamiltonians that

we have developed based on the Lie Algebras of the E2, E3 and E4 manifolds thus have counterparts in terms of the Klein Gordon equations derived using the spatial metrics underlying these manifolds.

In the case of spin, the situation is more complicated. The total angular momentum operator (including spin) for E3 (3.15) can be derived via the Klein Gordon equation from a metric that includes an extra compact dimension with coordinate χ. If we form the Laplacian using the metric for E3 (s≠0) we obtain:

$$\hat{J}^2 = -\nabla^2_{\theta\phi\chi} = -\left(\nabla^2_{\theta\phi} - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial^2}{\partial\phi\partial\chi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\chi^2} \right). \tag{5.7}$$

The Klein Gordon equation based on this Laplacian has solutions that incorporate the E3 generalised spherical harmonics, discussed in Chapter Three, coupled to harmonic evolution over this third angular coordinate (Willard Miller, 1964):

$$\psi(\theta,\phi,\chi) = Y^l_{s,m}(\theta,\phi)\exp(is\chi). \tag{5.8}$$

Substituting (5.7) and (5.8) into the Klein Gordon equation and eliminating χ then recovers (3.15).

Thus, in the case of spin, the Euler angles in the wavefunction solutions of the total angular momentum operator can only be described on a manifold of higher dimensionality than the manifold describing position; exploring spin solutions to the Lie Algebra of E3 requires a position configuration space larger than the S^2 spatial manifold that we started with.

If we consider perturbations to the metric, these naturally correspond to perturbations to the Klein Gordon equation. Metric coefficients are real, but in principle it is possible to

introduce complex terms into the expanded Klein Gordon equation starting from a general real metric:

$$\nabla^2\Psi + \frac{1}{\sqrt{-g}}\left(\partial_0 g^{i0}\sqrt{-g}\partial_i\Psi + \partial_i g^{i0}\sqrt{-g}\partial_0\Psi + \partial_0 g^{00}\sqrt{-g}\partial_0\Psi\right) = m^2\Psi\,. \qquad (5.9)$$

Complex terms in (5.9) could be generated, for example, from the harmonic time evolution of Ψ combined either with time variations in the g^{00} element, or with spatially varying g^{0i} elements. Thus the introduction of complex terms into the Klein Gordon equation by modifications to the metric might provide a route for studying PT symmetric potentials against background metrics. The algebraic relationships between changes to the metric and the resulting perturbations to the Klein Gordon equation are however complicated.

Relationship of PT Symmetric Hamiltonians to Lie Algebras

By contrast, we have seen that a group theoretic approach can provide a practical framework for constructing PT symmetric Hamiltonians as perturbations to the free Hamiltonian for quantum states on some manifold. What is intriguing is that the Lie Algebra framework incorporates PT symmetric Hamiltonians just as naturally as Hermitian Hamiltonians.

The PT symmetric Hamiltonians that we have studied essentially fall into two categories. The first category incorporates Lie Algebra raising or lowering operators; these are associated with upper or lower triangular Hamiltonians and lead to unbounded regions of unbroken PT symmetry for all representations. The second category incorporates position or momentum operators (multiplied by imaginary parameters); these PT Symmetric Hamiltonians have limited regions of unbroken PT symmetry

(essentially just for zero spin representations). Outside these regions of unbroken PT symmetry this second category has complex energy eigenvalues.

These behaviours arise because the manifolds studied are highly symmetric and therefore have multiple degenerate basis eigenstates that are related by Parity and Time Reversal (complex conjugation) symmetries.

Just as there is a fundamental difference in a Lie Algebra between root and raising/lowering operators, the different corresponding categories of PT symmetric Hamiltonians have different characteristic modes of mixing basis eigenstates and only certain of these modes give rise to regions of unbroken PT symmetry.

A further important observation is that the question as to whether a given PT symmetric Hamiltonian has unbroken or broken PT symmetry does not always permit an absolute answer. Thus, while the reality of its trace is a necessary condition for a Hamiltonian to have a region of unbroken PT symmetry, it is not a sufficient condition. For some Hamiltonians, such as those incorporating directional potentials, the emergence of unbroken PT symmetry depends crucially on the irrep within which we are working. Specifically, while we have identified regions of unbroken PT symmetry for zero spin bosonic representations, we have so far been unable to identify such regions for fermionic representations perturbed by directional potentials. (The situation for non-zero spin bosonic representations depends on the orientation of their orbital and spin momenta relative to the potential).

Thus, it appears clear that the presence of a region of unbroken PT symmetry is not simply a property of a given Hamiltonian but rather a property of the interactions between the Hamiltonian and a chosen wavefunction representation.

We have not attempted generally to reconstruct the Hermitian versions of all the PT symmetric Hamiltonians with unbroken PT symmetry that we have studied, since we have been working within a truncated (and therefore inexact) matrix mechanics framework, which does not lend itself to inversion of the underlying analytic relationships. We can however note the examples given in (2.12) and (3.29) of PT symmetric Hamiltonians incorporating raising/lowering operators that are related by similarity transformations effected by diffeomorphisms (coordinate displacements in the complex plane) to Hermitian Hamiltonians.

The approach set out herein can in principle be generalised, by use of commutation and recursion relations, to determine the PT symmetric properties of the many potentials that can be constructed as linear combinations, or even power series, of the various elements of a Lie Algebra. These operator combinations may be Hermitian or PT symmetric or neither. If PT symmetric, they may or may not have regions of unbroken PT symmetry, depending on the structure of their Hamiltonian matrices.

Stable and Unstable Systems

The importance of real eigenvalue solutions to the Klein Gordon or Schrodinger equations is that they represent states that are harmonic and thus stable over time, rather than decaying or increasing exponentially. Furthermore, since real eigenvalues ensure unitary evolution, $|\Psi_0\rangle \rightarrow e^{iHt}|\Psi_0\rangle = U|\Psi_0\rangle$, where H is Hermitian and U is unitary, the norm of a state $\langle\psi|\psi\rangle$, which is usually interpreted as a probability, is preserved over time. We saw in Chapter One how to modify the inner product definition so that non-Hermitian Hamiltonians with real eigenvalues can be assigned norms that are preserved over time. However this is no longer the case once eigenvalues become

complex and in order to work consistently with complex eigenvalues, it would be necessary to introduce decay/growth correction factors into the inner product. Non-unitary evolution therefore would create complications in the interpretation of probabilities.

On the other hand, traditional quantum theory continues to offer little fundamental insight into the nature of the transition from one quantum state to another, giving rise to the well known measurement paradoxes (Penrose, 2004). Under unitary evolution, a state evolves without losing or gaining information; Schrödinger's cat can evolve only into a superposition of states and the question as to whether the cat is alive or dead can never be definitively resolved. With non-Hermitian matrices, we have a mechanism for non-unitary evolution, in which a system could lose or gain information by states decaying in an irreversible manner. The boundary in PT symmetric systems between regions of unbroken and broken PT symmetry is a boundary at which phase transitions to such non-unitary behaviour could occur.

Conclusions

In conclusion, this study of PT symmetric quantum mechanics on E2, E3 and E4 has identified different types of PT symmetric Hamiltonian. These include PT symmetric Hamiltonians that have unbounded regions of unbroken PT symmetry, where all eigenvalues are real, as well as PT symmetric Hamiltonians which have limited parametric regions of unbroken PT symmetry. In this later category, the existence of unbroken PT symmetry depends on the group representation as well as on the values of coupling parameters. In particular there appears to be a fundamental difference between the PT symmetric behaviour of bosonic zero spin and fermionic representations, with fermionic representations facing PT symmetry breaking for certain given forms of

Hamiltonian under which zero spin bosonic representations enjoy unbroken PT symmetry.

Also, there appear to be fundamental differences between the interactions of raising and lowering operators, which are non-Hermitian but PT symmetric, and those of root operators, such as momentum or position operators, which can be used to construct either Hermitian or PT symmetric Hamiltonians.

In all cases, the interplay between the structure of a Hamiltonian, in terms of its constituent Lie Algebra operators, and the irreps of the relevant Euclidean group determines the pattern of PT symmetry breaking.

Further Work

The E2, E3, and E4 manifolds, being rotationally symmetric and compact, represent a small subset of spatial manifolds, and their natural embedding in spacetime takes the form of metrics described by (5.4), which in turn represent a small subset of spacetime manifolds. The prerequisites of the approach developed herein, which involves building upon the Lie Algebras of position and momentum operators over a manifold, are a high degree of symmetry. There appears no reason a priori, therefore, why it should not be possible to apply this approach to explore the PT symmetric quantum theory of open and/or hyperbolic rather than compact spatial manifolds and also to study manifolds where time and space are coupled in a non-trivial manner.

Also, this study of non-zero spin representations has been limited to E3. It is clear that the introduction of spin has significant consequences for the PT symmetric quantum behaviour of E3 and we cannot entirely predict the behaviours of non-zero spin representations ($W^2 \neq 0$) of E4 (and higher dimensioned manifolds) from those of E3.

Thus it should be interesting, not only to extend the E4 work to cover non-zero spin representations, but also to study the PT symmetry of spacetimes with a high degree of symmetry such as Anti deSitter and deSitter and also to extend the analysis to spacetime manifolds whose metrics incorporate spin, such as Kerr-Newman spacetimes.

Appendices

Normalisation of E4 Eigenstates

The normalisation of the top state in any E4 l-multiplet is given by:

$$\langle l,0,l|l,0,l\rangle = |N_l|^2 \int \underbrace{\sin^2\psi \sin\theta d\psi d\theta d\phi}_{volume\ element} \left(\cos^2\psi + \cos^2\theta \sin^2\psi\right)^l = |N_l|^2 \frac{2\pi^2}{1+l} \equiv 1.$$

So: $$\boxed{N_l = \frac{1}{\pi}\sqrt{\frac{1+l}{2}}}.$$

We can normalise any other state in the multiplet relative to the top state using the E4 commutation relations as follows:

$$|l,m,n\rangle = N_{lmn}\ \hat{J}_{\alpha-}^{(l-m-n)/2}\ \hat{J}_{\beta-}^{(l+m-n)/2}|l,0,l\rangle,$$

$$\langle l,m,n||l,m,n\rangle = |N_{lmn}|^2 \langle l,0,l|\hat{J}_{\beta+}^{(l+m-n)/2}\hat{J}_{\alpha+}^{(l-m-n)/2}\ \hat{J}_{\alpha-}^{(l-m-n)/2}\ \hat{J}_{\beta-}^{(l+m-n)/2}|l,0,l\rangle$$

$$= |N_{lmn}|^2 \langle l,0,l|\hat{J}_{\alpha+}^{(l-m-n)/2}\ \hat{J}_{\alpha-}^{(l-m-n)/2}\ \hat{J}_{\beta+}^{(l+m-n)/2}\hat{J}_{\beta-}^{(l+m-n)/2}|l,0,l\rangle.$$

From the *Lemma*, setting r=0 :

$$\hat{J}_{\beta+}^{p}\ \hat{J}_{\beta-}^{p}|l,0,l\rangle = \hat{J}_{\alpha+}^{p}\ \hat{J}_{\alpha-}^{p}|l,0,l\rangle = 4^p \frac{p!l!}{(l-p)!}|l,0,l\rangle.$$

So: $$\langle l,m,n||l,m,n\rangle = |N_{lmn}|^2 \langle l,0,l|4^{l-n}\frac{(l-m-n)/2!l!}{(l+m+n)/2!}\ \frac{(l+m-n)/2!l!}{(l-m+n)/2!}|l,0,l\rangle \equiv 1.$$

So: $$|N_{lmn}|^2 = \frac{1}{4^{l-n}}\frac{(l+m+n)/2!(l-m+n)/2!}{(l-m-n)/2!(l+m-n)/2!l!l!}.$$

So: $$\boxed{N_{lmn} = \frac{1}{2^{l-n}l!}\sqrt{\frac{(l+m+n)/2!(l-m+n)/2!}{(l-m-n)/2!(l+m-n)/2!}}}.$$

Lemma :

Consider: $\hat{J}_{\alpha+}^{p-r}\ \hat{J}_{\alpha-}^{p}|l,0,l\rangle$

$$= \hat{J}_{\alpha+}^{p-r-1}\hat{J}_{\alpha+}\ \hat{J}_{\alpha-}\hat{J}_{\alpha-}^{p-1}|l,0,l\rangle$$

$$= \hat{J}_{\alpha+}^{p-r-1}\left[4\left(\hat{\mathrm{J}}_{uv}+\hat{\mathrm{J}}_{xy}\right)\hat{J}_{\alpha-}^{p-1} + \hat{J}_{\alpha-}4\left(\hat{\mathrm{J}}_{uv}+\hat{\mathrm{J}}_{xy}\right)\hat{J}_{\alpha-}^{p-2} + \ldots\hat{J}_{\alpha-}^{p-1}4\left(\hat{\mathrm{J}}_{uv}+\hat{\mathrm{J}}_{xy}\right)\right]|l,0,l\rangle$$

$$= \hat{J}_{\alpha+}^{p-r-1}\left[4\left(l-2(p-1)\right)\hat{J}_{\alpha-}^{p-1} + \hat{J}_{\alpha-}4\left(l-2(p-2)\right)\hat{J}_{\alpha-}^{p-2} + \ldots\hat{J}_{\alpha-}^{p-1}4(l+0)\right]|l,0,l\rangle$$

$$= 4p\left[l-(p-1)\right]\hat{J}_{\alpha+}^{p-r-1}\hat{J}_{\alpha-}^{p-1}|l,0,l\rangle.$$

So: $$\boxed{\hat{J}_{\alpha+}^{p-r}\ \hat{J}_{\alpha-}^{p}|l,0,l\rangle = 4^{p-r}\frac{p!}{r!}\frac{(l-r)!}{(l-p)!}|l,0,l\rangle}.$$

A similar result applies for $\hat{J}_{\beta+}^{p-r}\ \hat{J}_{\beta-}^{p}$.

Calculation of E4 Matrix Elements

We wish to obtain the matrix element: $\langle l',m',n'|\hat{u}|l,m,n\rangle = \langle l',m',n'|\frac{1}{2}(U_+ + U_-)|l,m,n\rangle$.

Recall: $\left[\hat{J}_{\alpha-},U_-\right]=\left[\hat{J}_{\beta-},U_-\right]=\left[\hat{J}_{\beta+},U_+\right]=\left[\hat{J}_{\alpha+},U_+\right]=\left[\hat{J}_{\alpha\pm},\hat{J}_{\beta\pm}\right]=0,$ where

U_+ raises n leaving m fixed, $\hat{J}_{\alpha+}$ raises both n and m, and $\hat{J}_{\beta+}$ raises n while lowering m.

So: $|l,m,n\rangle = N_{lmn}\hat{J}_{\alpha-}^{(l-m-n)/2}\ \hat{J}_{\beta-}^{(l+m-n)/2}|l,0,l\rangle,$

$\langle l',m',n'|U_+|l,m,n\rangle = N_{l'm'n'}N_{lmn}\langle l',0,l'|\hat{J}_{\alpha+}^{(l'-m'-n')/2}\ \hat{J}_{\beta+}^{(l'+m'-n')/2}U_+\hat{J}_{\alpha-}^{(l-m-n)/2}\ \hat{J}_{\beta-}^{(l+m-n)/2}|l,0,l\rangle$

$= N_{l'm'n'}N_{lmn}\langle l',0,l'|U_+\hat{J}_{\alpha+}^{(l'-m'-n')/2}\hat{J}_{\alpha-}^{(l-m-n)/2}\ \hat{J}_{\beta+}^{(l'+m'-n')/2}\ \hat{J}_{\beta-}^{(l+m-n)/2}|l,0,l\rangle.$

For a non-zero matrix element we require $n' = n+1$ and $m' = m$ (action of U_+).

Consider the numerical bounds on l for non-anihilation of top state $|l,0,l\rangle$ by $\hat{J}_{\alpha+}$and $\hat{J}_{\beta+}$.

We require : $l+m-n \geq l'+m'-n',$

And : $l-m-n \geq l'-m'-n'.$

So: $l' \leq l+1.$

So: $l' = l+1-2r,$ where r is an integer ≥0.

So: $\langle l',m',n'|U_+|l,m,n\rangle$

$= N_{l'm'n'}N_{lmn}\langle l+1-2r,0,l+1-2r|U_+\hat{J}_{\alpha+}^{(l-m-n)/2-r}\hat{J}_{\alpha-}^{(l-m-n)/2}\ \hat{J}_{\beta+}^{(l+m-n)/2-r}\ \hat{J}_{\beta-}^{(l+m-n)/2}|l,0,l\rangle.$

Now : $N_{lmn} = \frac{1}{2^{l-n}l!}\sqrt{\frac{(l+m+n)/2!(l-m+n)/2!}{(l-m-n)/2!(l+m-n)/2!}}.$

And: $\hat{J}_{\beta+}^{p-r}\ \hat{J}_{\beta-}^{p}|l,0,l\rangle = 4^{p-r}\frac{p!}{r!}\frac{(l-r)!}{(l-p)!}.$

So: $\langle l',m',n'|U_+|l,m,n\rangle = F(l,m,n,r)\langle l+1-2r,0,l+1-2r|U_+\hat{J}_{\alpha-}^{r}\ \hat{J}_{\beta-}^{r}|l,0,l\rangle.$

Where: $F(l,m,n,r) = N_{l'm'n'}N_{lmn}4^{l-n-2r}\frac{(l-r)!(l-r)!(l-m-n)/2!(l+m-n)/2!}{r!r!(l+m+n)/2!(l-m+n)/2!}.$

Now, since $\hat{J}_{\alpha-}$ anihilates the top state on the left and $\left[\hat{J}_{\alpha-},\hat{J}_{\beta-}\right]=0$:

$\langle l+1-2r,0,l+1-2r|U_+\hat{J}_{\alpha-}^{r}\ \hat{J}_{\beta-}^{r}|l,0,l\rangle$

$= \langle l+1-2r,0,l+1-2r|\left[U_+,\hat{J}_{\alpha-}\right]\hat{J}_{\beta-}\hat{J}_{\alpha-}^{r-1}\ \hat{J}_{\beta-}^{r-1}|l,0,l\rangle$

$= 2\langle l+1-2r,0,l+1-2r|\left[X_-,\hat{J}_{\beta-}\right]\hat{J}_{\alpha-}^{r-1}\ \hat{J}_{\beta-}^{r-1}|l,0,l\rangle$

$= 4\langle l+1-2r,0,l+1-2r|U_-\hat{J}_{\alpha-}^{r-1}\ \hat{J}_{\beta-}^{r-1}|l,0,l\rangle$

$= 0$ for r>1, since $\left[\hat{J}_{\alpha-},U_-\right]=0$ and the $\hat{J}_{\alpha-}$ anihilate on the left.

We are left with r=0 and r=1 as the only possibilities.

Now : $F(l,m,n,0)=\frac{1}{2(l+1)}\sqrt{(l+m+n+2)(l-m+n+2)}.$

And : $F(l,m,n,1)=\frac{1}{8l}\sqrt{(l+m-n)(l-m-n)}.$

So we have for r=0 and r=1 respectively:

$$\langle l+1,m,n+1|U_+|l,m,n\rangle=\frac{1}{2(l+1)}\sqrt{(l+m+n+2)(l-m+n+2)}\,\langle l+1,0,l+1|U_+|l,0,l\rangle,$$

$$\langle l-1,m,n+1|U_+|l,m,n\rangle=\frac{1}{2l}\sqrt{(l+m-n)(l-m-n)}\,\langle l-1,0,l-1|U_-|l,0,l\rangle.$$

Taking the Hermitian conjugate for r=0 and letting $l\to$ l-1, $n\to n-1$:

$$\langle l-1,m,n-1|U_-|l,m,n\rangle=\frac{1}{2l}\sqrt{(l+m+n)(l-m+n)}\,\langle l-1,0,l-1|U_-|l,0,l\rangle.$$

Taking the Hermitian conjugate for r=1 and letting $l\to$ l+1, $n\to n-1$:

$$\langle l+1,m,n-1|U_-|l,m,n\rangle=\frac{1}{2(l+1)}\sqrt{(l+m-n+2)(l-m-n+2)}\,\langle l+1,0,l+1|U_+|l,0,l\rangle.$$

We can evaluate the matrix elements with the top state analytically:

$$\begin{aligned}
&\langle l+1,0,l+1|U_+|l,0,l\rangle\\
&\quad=N_{l+1}N_l\int vol\left(\cos\psi-i\cos\theta\sin\psi\right)^{l+1}\underbrace{\left(\cos\psi+i\cos\theta\sin\psi\right)}_{U_+}\left(\cos\psi+i\cos\theta\sin\psi\right)^{l}\\
&\quad=N_{l+1}N_l\int vol\left(\cos^2\psi+\cos^2\theta\sin^2\psi\right)^{l+1}\\
&\quad=\frac{1}{\pi}\sqrt{\frac{2+l}{2}}\,\frac{1}{\pi}\sqrt{\frac{1+l}{2}}\,\frac{2\pi^2}{2+l}\\
&\quad=\sqrt{\frac{l+1}{l+2}}.
\end{aligned}$$

Similarly:

$$\langle l-1,0,l-11|U_-|l,0,l\rangle=\sqrt{\frac{l}{l+1}},$$

$$\langle l-1,0,l-1|U_-|l,0,l\rangle=\sqrt{\frac{l}{l+1}},$$

$$\langle l+1,0,l+1|U_+|l,0,l\rangle=\sqrt{\frac{l+1}{l+2}}.$$

So we can derive matrix elements for the raising and lowering operators:

$$\langle l+1,m,n+1|U_+|l,m,n\rangle = \frac{1}{2}\sqrt{\frac{(l+m+n+2)(l-m+n+2)}{(l+1)(l+2)}},$$

$$\langle l-1,m,n+1|U_+|l,m,n\rangle = \frac{1}{2}\sqrt{\frac{(l+m-n)(l-m-n)}{l(l+1)}},$$

$$\langle l-1,m,n-1|U_-|l,m,n\rangle = \frac{1}{2}\sqrt{\frac{(l+m+n)(l-m+n)}{l(l+1)}},$$

$$\langle l+1,m,n-1|U_-|l,m,n\rangle = \frac{1}{2}\sqrt{\frac{(l+m-n+2)(l-m-n+2)}{(l+1)(l+2)}}.$$

These combine to give matrix elements for the position operator:

$$\langle l+1,m,n+1|\hat{u}|l,m,n\rangle = \frac{1}{4}\sqrt{\frac{(l+m+n+2)(l-m+n+2)}{(l+1)(l+2)}},$$

$$\langle l-1,m,n+1|\hat{u}|l,m,n\rangle = \frac{1}{4}\sqrt{\frac{(l+m-n)(l-m-n)}{l(l+1)}},$$

$$\langle l-1,m,n-1|\hat{u}|l,m,n\rangle = \frac{1}{4}\sqrt{\frac{(l+m+n)(l-m+n)}{l(l+1)}},$$

$$\langle l+1,m,n-1|\hat{u}|l,m,n\rangle = \frac{1}{4}\sqrt{\frac{(l+m-n+2)(l-m-n+2)}{(l+1)(l+2)}}.$$

It should be noted that we need to specify the phases of the various eigenstates and that a conventional choice, consistently applied, will determine the signs of the various matrix elements.

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